CSC 411: Introduction to Machine Learning Lecture 5: Ensembles II

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Boosting

- Recall that an ensemble is a set of predictors whose individual decisions are combined in some way to classify new examples.
- (Previous lecture) Bagging: Train classifiers independently on random subsets of the training data.
- (This lecture) **Boosting**: Train classifiers sequentially, each time focusing on training data points that were previously misclassified.
- Let's start with the concepts of weighted training sets and weak learner/classifier (or base classifiers).

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• Usually require each $w^{(n)} > 0$ and $\sum_{n=1}^{N} w^{(n)} = 1$

Weak Learner/Classifier

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Weak Learner/Classifier

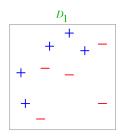
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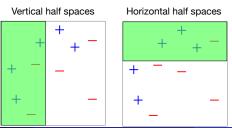
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- We are interested in weak learners that are computationally efficient.
 - Decision trees
 - ▶ Even simpler: Decision Stump: A decision tree with only a single split

[Formal definition of weak learnability has quantifies such as "for any distribution over data" and the requirement that its guarantee holds only probabilistically.]

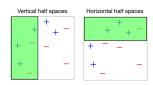
Weak Classifiers



These weak classifiers, which are decision stumps, consist of the set of horizontal and vertical half spaces.



Weak Classifiers



• A single weak classifier is not capable of making the training error very small. It only perform slightly better than chance, i.e., the error of classifier h according to the given weights $\{w^{(1)},\ldots,w^{(N)}\}$ (with $\sum_{n=1}^N w^{(n)}=1$ and $w^{(n)}\geq 0$)

$$\mathsf{err} = \sum_{n=1}^{N} w^{(n)} \mathbb{I}[h(\mathbf{x}^{(n)}) \neq t^{(n)}]$$

is at most $\frac{1}{2} - \gamma$ for some $\gamma > 0$.

 Can we combine a set of weak classifiers in order to make a better ensemble of classifiers?

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- The weak learner needs to minimize weighted error.
- AdaBoost reduces bias by making each classifier focus on previous mistakes.

• Input: Data $\mathcal{D}_N = \{\mathbf{x}^{(n)}, t^{(n)}\}_{n=1}^N$, weak classifier WeakLearn (a classification procedure that return a classifier from base hypothesis space \mathcal{H} with $h: \mathbf{x} \to \{-1, +1\}$ for $h \in \mathcal{H}$), number of iterations T

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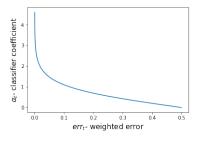
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• Return $H(\mathbf{x}) = \operatorname{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(\mathbf{x})\right)$

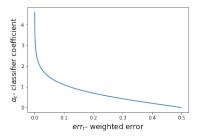
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• Recall: $H(\mathbf{x}) = \operatorname{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(\mathbf{x})\right)$ where $\alpha_t = \frac{1}{2} \log \frac{1 - \operatorname{err}_t}{\operatorname{err}_t}$



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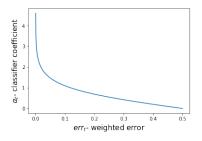
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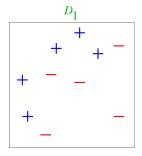
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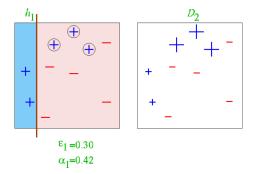


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- Also: $w^{(n)} \leftarrow w^{(n)} \exp \left(2\alpha_t \mathbb{I}\{h_t(\mathbf{x}^{(n)}) \neq t^{(n)}\}\right)$
 - ▶ If $err_t \approx 0$, α_t high so misclassified examples more attention
 - ▶ If $err_t \approx 0.5$, α_t low so misclassified examples are not emphasized

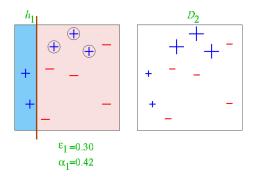
• Training data



• Round 1

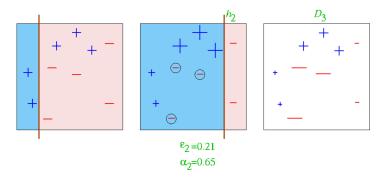


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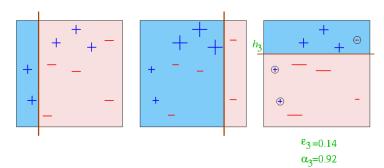
$$\mathbf{w} = \left(\frac{1}{10}, \dots, \frac{1}{10}\right) \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_1 = \frac{\sum_{n=1}^{10} w^{(n)} \mathbb{I}[h_1(\mathbf{x}^{(n)}) \neq t^{(n)}]}{\sum_{n=1}^{10} w^{(n)}} = \frac{3}{10}$$
$$\Rightarrow \alpha_1 = \frac{1}{2} \log \frac{1 - \text{err}_1}{\text{err}_1} = \frac{1}{2} \log (\frac{1}{0.3} - 1) \approx 0.42 \Rightarrow H(\mathbf{x}) = \text{sign} \left(\alpha_1 h_1(\mathbf{x})\right)$$

Round 2



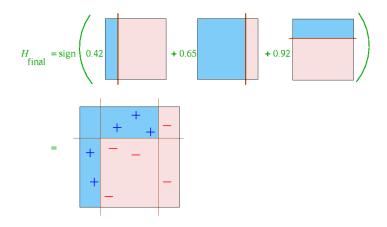
$$\mathbf{w} \leftarrow \text{new weights} \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_2 = \frac{\sum_{n=1}^{10} w^{(n)} \mathbb{I}\{h_2(\mathbf{x}^{(n)}) \neq t^{(n)}\}}{\sum_{n=1}^{10} w^{(n)}} = 0.21$$
$$\Rightarrow \alpha_2 = \frac{1}{2} \log \frac{1 - \text{err}_2}{\text{err}_2} = \frac{1}{2} \log (\frac{1}{0.21} - 1) \approx 0.66 \Rightarrow H(\mathbf{x}) = \text{sign} (\alpha_1 h_1(\mathbf{x}) + \alpha_2 h_2(\mathbf{x}))$$

Round 3

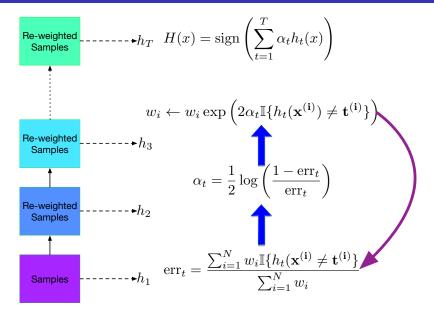


$$\begin{aligned} \mathbf{w} \leftarrow \text{new weights} \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_3 &= \frac{\sum_{n=1}^{10} w^{(n)} \mathbb{I}\{h_3(\mathbf{x}^{(n)}) \neq t^{(n)}\}}{\sum_{i=1}^{10} w^{(n)}} = 0.14 \\ \Rightarrow &\alpha_3 &= \frac{1}{2} \log \frac{1 - \text{err}_3}{\text{err}_3} &= \frac{1}{2} \log (\frac{1}{0.14} - 1) \approx 0.91 \Rightarrow H(\mathbf{x}) = \text{sign} \left(\alpha_1 h_1(\mathbf{x}) + \alpha_2 h_2(\mathbf{x}) + \alpha_3 h_3(\mathbf{x})\right) \end{aligned}$$

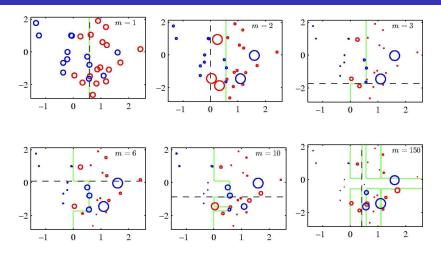
Final classifier



AdaBoost Algorithm



AdaBoost Example



• Each figure shows the number m of base learners trained so far, the decision of the most recent learner (dashed black), and the boundary of the ensemble (green)

AdaBoost Minimizes the Training Error

Theorem

Assume that at each iteration of AdaBoost the WeakLearn returns a hypothesis with error $\operatorname{err}_t \leq \frac{1}{2} - \gamma$ for all $t = 1, \ldots, T$ with $\gamma > 0$. The training error of the output hypothesis $H(\mathbf{x}) = \operatorname{sign}\left(\sum_{t=1}^T \alpha_t h_t(\mathbf{x})\right)$ is at most

$$L_N(H) = \frac{1}{N} \sum_{i=1}^N \mathbb{I}\{H(\mathbf{x}^{(i)}) \neq t^{(i)})\} \leq \exp\left(-2\gamma^2 T\right).$$

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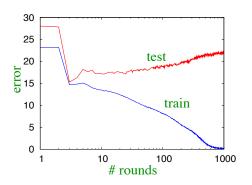
- \bullet This is under the simplifying assumption that each weak learner is $\gamma\text{-better}$ than a random predictor.
- Analyzing the convergence of AdaBoost is generally difficult.

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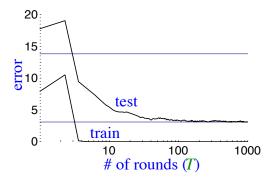
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- We expect more complex classifiers overfit.
- If one runs AdaBoost long enough, it can in fact overfit.

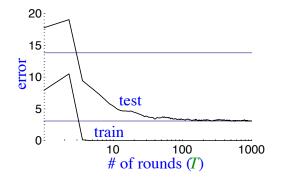


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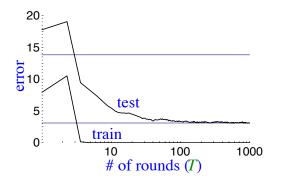


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- How does that happen?
- We will provide an alternative viewpoint on AdaBoost later in the course.

[Slide credit: Robert Shapire's Slides, http://www.cs.princeton.edu/courses/archive/spring12/cos598A/schedule.html]

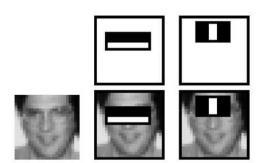
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 - Change loss function for weak learners: false positives less costly than misses
 - ► Smart way to do inference in real-time (in 2001 hardware)







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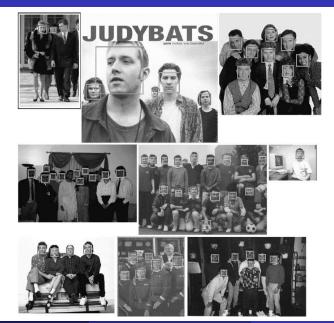


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 - ► There is a neat trick for computing the total intensity in a rectangle in a few operations.
 - So it is easy to evaluate a huge number of base classifiers and they are very fast at runtime.
 - The algorithm adds classifiers greedily based on their quality on the weighted training cases
 - ► Each classifier uses just one feature

AdaBoost Face Detection Results



Summary

- Boosting reduces bias by generating an ensemble of weak classifiers.
- Each classifier is trained to reduce errors of previous ensemble.
- It is quite resilient to overfitting, though it can overfit.
- We will later provide a loss minimization viewpoint to AdaBoost. It allows us to derive other boosting algorithms for regression, ranking, etc.

Ensembles Recap

- Ensembles combine classifiers to improve performance
- Boosting
 - Reduces bias
 - Increases variance (large ensemble can cause overfitting)
 - Sequential
 - ▶ High dependency between ensemble elements
- Bagging
 - Reduces variance (large ensemble can't cause overfitting)
 - Bias is not changed (much)
 - Parallel
 - Want to minimize correlation between ensemble elements.
- Next Lecture: Linear Regression