The adaptable chromatic number and the chromatic number

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Abstract

We prove that the adaptable chromatic number of a graph is at least asymptotic to the square root of the chromatic number. This is best possible.

Consider a graph G where each edge of G is assigned a colour from $\{1, ..., k\}$ (this is not necessarily a proper edge colouring). A *k-adapted colouring* is an assignment of colours from $\{1, ..., k\}$ to the vertices of G such that there is no edge with the same colour as both of its endpoints. In other words: in conventional graph colouring, each edge forbids its endpoints from both receiving the same colour, while in adaptable colouring, each edge is given one particular colour which it forbids its endpoints from both receiving. The *adaptable chromatic number*, $\chi_a(G)$, of a graph G is the minimum value of k such that every k-edge colouring of G can be completed into a k-adapted colouring.

To be clear: an adapted colouring might not be proper. Eg. two adjacent vertices u, v can both receive the colour Red if the edge uv has a colour other than Red.

It is not surprising that this natural variation on graph colouring has arisen in a variety of settings. Hell and Zhu[9] were the first to use the terminology *adaptable colouring*. But it was introduced independently as *split colourings*[4], *emulsive colourings*[2] and *chromatic capacity*[1].

Greene[7] was the first to conjecture the adaptable chromatic number grows with the chromatic number. He suggested that possibly $\chi_a(G)$ is always as high as $\Theta(\sqrt{\chi(G)})$, noting that this would be best possible up to the multiplicative constant as Erdős and Gyárfás[4] had shown that $(1 + o(1))\sqrt{n} \leq \chi_a(K_n) \leq \sqrt{2n}$. Huizenga[10] proved that the conjecture holds for almost all graphs. Zhou[14] proved the conjecture for every graph, showing that $\chi_a(G) \geq \Theta(\log\log\chi(G))$. Other work on adaptable colouring can be found in [3, 5, 6, 8, 11, 12, 13, 15].

Here, we give a very short proof of a bound that is tight up to second order terms:

Theorem 1 $\chi_a(G) \geq (1 + o(1))\sqrt{\chi(G)}$.

The tightness follows from the main theorem of [12] which showed that if G has maximum degree Δ then $\chi_a(G) \leq (1 + o(1))\sqrt{\Delta}$, thus improving the constant from the bound in [4] to

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$\chi_a(K_n) \leq (1+o(1))\sqrt{n}$. In fact, the bound from [12] applies to the list colouring version of adapted colouring and so Theorem 1 is also a tight bound on the *adaptable list chromatic number*.

We use a very nice idea from Zhou[14], whose approach was: take a proper q-vertex colouring of a graph G with $\chi(G) = q$. Then $f(q)$-colour the edges of G according to a specific pattern based on that vertex colouring. Suppose that this can be completed to a $f(q)$-adapted colouring, and then use that adapted colouring to obtain a proper $(q-1)$-vertex colouring of G. This contradiction shows that if $\chi(G) = q$ then $\chi_a(G) > f(q)$. His edge colouring followed a recursive pattern which led to $f(q) = \Theta(\log\log q)$; our improvement comes from taking a more balanced pattern.

Given a (not necessarily proper) edge colouring of a graph K_n, we define H_i to be the subgraph formed by removing every edge of colour i from K_n.

Lemma 2 *If K_n has a k-edge colouring for which $\sum_{i=1}^{k} \chi(H_i) < n$, then every graph G with $\chi(G) = n$ has $\chi_a(G) > k$.*

Proof Let ψ denote the k-edge-colouring of K_n. Consider any proper n-colouring σ of G, and identify the colours $\{1, ..., n\}$ with the vertices of K_n. We k-colour the edges of G according to ψ; specifically, we give each edge uv the colour $\psi(\sigma(u)\sigma(v))$.

If $\chi_a(G) \leq k$ then this edge-colouring can be completed to a k-adapted colouring; let S_i be the vertices of colour i in the adapted colouring for $i = 1, ..., k$. Thus there is no edge of colour i between any two vertices in S_i. It follows that we can properly $\chi(H_i)$-colour the vertices of S_i: give each $v \in S_i$ the colour that $\sigma(v)$ gets in a specific proper $\chi(H_i)$-colouring of H_i. If $u, v \in S_i$ get the same colour, then $\sigma(u), \sigma(v)$ have the same colour in the colouring of H_i. So either (a) $\sigma(u) = \sigma(v)$ or (b) the edge $\sigma(u), \sigma(v)$ is not in H_i and hence has colour i in ψ. In case (a), u, v are in the same colour class of σ and so are not adjacent. In case (b), if there is an edge between u, v then it has colour i which contradicts the fact that $u, v \in S_i$ in the adapted colouring. So if $u, v \in S_i$ get the same colour then they are not adjacent in G; i.e. this is a proper $\chi(H_i)$-colouring of S_i.

Using disjoint sets of $\chi(H_i)$ colours for each S_i, we obtain a proper colouring of G using $\sum_{i=1}^{k} \chi(H_i) < n = \chi(G)$ colours. This contradiction implies that $\chi_a(G) > k$. □

We obtain our edge-colouring of K_n using the following well-known construction:

Lemma 3 *If there is a projective plane of order r, then we can $(r+1)$-colour the edges of K_{r^2} such that each colour class is the edge-set of the union of r vertex-disjoint copies of K_r.*

Proof Let $y_1, ..., y_{r+1}$ be the points of a line, L, in the projective plane. Let u, v be any two points not on that line. We assign the colour i to the pair $\{u, v\}$ if the line through u, v intersects L at y_i. Since there are r^2 vertices not on L, and each pair lies on a line that intersects L in a unique point, this provides an edge-colouring of K_{r^2}. Each colour i is assigned to the edges of the r copies of K_r formed by each of the other r lines through y_i. □

Remark: Erdős and Gyárfás[4] used a slightly more complicated construction to colour some of the edges of K_{r^2+r+1} so that each colour forms r copies of K_r and one copy of K_{r+1}; using that colouring would provide a small improvement in the $o(1)$ term of our main theorem.

Proof of Theorem 1: We begin with the case $n = r^2$ where r is the square of a prime power. So there is a projective plane of order r, and thus Lemma 3 yields an $(r+1)$-colouring of the edges of K_n. We change this to an $(r-1)$-colouring by arbitrarily recolouring all edges with colour r or $r+1$. This yields subgraphs $H_1, ..., H_{r-1}$ which are easily seen to be properly r-colourable. Therefore, $\sum_{i=1}^{k} \chi(H_i) \leq r(r-1) < n$, and so Lemma 2 shows that if $\chi(G) = n$ then $\chi_a(G) \geq r = \sqrt{n}$. Now the density of the primes implies that the statement of the theorem holds for all n, where the $o(1)$ term depends on how much higher $\chi(G)$ is than the square of a prime power. $\square$

References

[1] A. Archer. *On the upper chromatic numbers of the reals.* Discrete Math, **214** (2000), 65-75.

[2] M. Cochand, P. Duchet. *A few remarks on orientations of graphs and Ramsey theory.* in: G. Halasz, V. T. Sos (Eds.), Irregularities of Partitions, Algorithms and Combinatorics, Vol. 8, Springer, Berlin (1989), 39-46.

[3] M. Cochand and G. Karolyi. *On a graph coloring problem.* Disc. Math. **194** (1999), 249-252.

[4] P. Erdős and A. Gyárfás. *Split and balanced colorings of complete graphs.* Discrete Math **200** (1999), 79-86.

[5] L. Esperet, M. Montassier and X. Zhu. *Adapted list colouring of planar graphs.* J. Graph Th. **62** (2007), 127 - 138.

[6] Z. Füredi and R. Ramamurthi. *On splittable colorings of graphs and hypergraphs.* J. Graph Theory **40** (2002), 226237.

[7] J.E. Greene. *Chromatic capacities of graphs and hypergraphs.* Disc. Math. **281** (2004), 197-207.

[8] P. Hell, Z. Pan, T. Wong and X. Zhu. *Adaptable colouring of graph products.* Disc. Math. **309** (2009), 6153 - 6159.

[9] P. Hell and X. Zhu. *Adaptable chromatic number of graphs.* European J. Combinatorics **29(4)** (2008), 912-921.

[10] J. Huizenga. *Chromatic capacity and graph operations.* Disc. Math. **308** (2008), 21342148.

[11] A. Kostochka and X. Zhu. *Adapted list coloring of graphs and hypergraphs.* SIAM J. Discrete Math **22** (2008), 398 - 408.

[12] M. Molloy and G. Thron. *An asymptotically tight bound on the adaptable chromatic number.* J. Graph Theory **71** (2012), 331-351.

[13] M. Molloy and G. Thron. *The adaptable choosability number grows with the choosability number.* Disc. Math. **311** (2011), 2268-2271.

[14] B. Zhou. *A lower bound for the chromatic capacity in terms of the chromatic number of a graph.* Disc. Math. **313** (2013) 2146 - 2149.

[15] B. Zhou. *On a construction of graphs with high chromatic capacity and large girth.* Disc. Math. **310** (2010), 2452 - 2454.