The adaptable choosability number grows with the choosability number

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Abstract

The *adaptable choosability number* of a multigraph G, denoted $ch_a(G)$, is the smallest integer k such that every edge labeling of G and assignment of lists of size k to the vertices of G permits a list coloring of G in which no edge $e = uv$ has both u and v colored with the label of e. We show that ch_a grows with ch, i.e. there is a function f tending to infinity such that $ch_a(G) \geq f(ch(G))$.

Keywords: adaptable coloring, list coloring

1. Introduction

Hell and Zhu introduced the adaptable chromatic number in [11]. Given a multigraph whose edges are labeled from $[k] = \{1, 2, \ldots, k\}$, the goal is to color the vertices with colors from $[k]$ so that there is no edge $e = uv$ such that u and v are both colored with the label of e. A vertex coloring which satisfies this property is called an *adaptable vertex coloring*. The *adaptable chromatic number* of a graph G, denoted $\chi_a(G)$, is the minimum number k such that *every* edge labeling of G from $[k]$ permits an adaptable vertex coloring from $[k]$. It has been studied in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] (in some cases by a different name).

Note that every proper vertex coloring of a graph G is an adaptable vertex coloring for any edge labeling and thus $\chi_a(G) \leq \chi(G)$. The inequality is tight as there are infinite families of graphs where $\chi_a(G) = \chi(G)$ [10, 11]. These parameters can also be far apart as there are infinite families of graphs where $\chi_a(G) = \Theta\left(\sqrt{\chi(G)}\right)$ (for example, the complete graph [4]). This brings us

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to the following question proposed by Hell and Zhu in [11].

Question. Is there a function f tending to infinity such that $\chi_a(G) \geq f(\chi(G))$?

As far as we know, the answer may be 'yes' with $f(k) = \Theta\left(\sqrt{k}\right)$; i.e. the complete graph may be asymptotically extremal.

In this paper, we study adaptable list coloring, which is defined naturally in [12]: Given a multigraph G, the *adaptable choosability number*, denoted $ch_a(G)$, is the minimum number k such that every edge labeling of G and assignment to each vertex v of a list $L(v)$ of size k, there is an adaptable coloring of G from these lists. As with χ_a, it is trivial that $ch_a(G) \leq ch(G)$, where $ch(G)$ is the choosability number. We answer the list coloring version of Hell and Zhu's question.

Theorem 1.1. *There is a function h tending to infinity such that $ch_a(G) \geq h(ch(G))$.*

Our proof obtains $h(k) = \Theta\left(log^{1/5}k\right)$, but we made no effort to optimize it. As far as we know, we can have $h(k) = \Theta(\sqrt{k})$. We know, however, that $h(k) = O(\sqrt{k})$ since, like with χ_a, the complete graph has $ch_a(K_n) = \Theta\left(\sqrt{ch(K_n)}\right)$ [12, 14].

The proof of the theorem uses a probabilistic approach and takes advantage of the Chernoff bound [3]. Instead of using the original statement we use the (weaker) version found in [13].

Chernoff Bound. *For any $0 \leq t \leq np$,*

$$\Pr\left(|BIN(n,p) - np| > t\right) < 2e^{-\frac{t^2}{3np}},$$

where $BIN(n,p)$ is the sum of n independent variables, each equal to 1 with probability p and 0 otherwise.

2. Proof of Main Theorem

The proof of Theorem 1.1 closely follows the approach taken by Alon in [1] for a similar result on normal list coloring.

We start by proving the following theorem, where $\delta(G)$ is the minimum degree of G.

Theorem 2.1. *There is a function g tending to infinity such that if H is a bipartite graph satisfying $\delta(H) \geq d$ then $ch_a(H) \geq g(d)$.*

Theorem 1.1 easily follows from Theorem 2.1.

Proof of Theorem 1.1. We use the following two well known and easily proved facts:

(i) If $\delta(G) \geq d$, G has a bipartite subgraph with minimum degree at least $\frac{d}{2}$. This can be seen by taking a spanning bipartite subgraph of G with the maximum number of edges.

(ii) If $ch(G) \geq k$, then it has a subgraph of minimum degree at least $k-1$. This can be seen by taking a graph where every subgraph has minimum degree at most $k-2$ and iteratively coloring the vertex with minimum degree and then removing it from the graph.

Therefore, the function $h(k) = g\left(\frac{k-1}{2}\right)$ satisfies the desired properties. □

Note that Fact (ii) holds for the coloring number (the maximum of $\delta(H)+1$ over all subgraphs H of G) as well, so Theorem 1.1 can be strengthened to show that ch_a grows with the coloring number.

To prove Theorem 2.1, consider any bipartite graph H with bipartition (A, B) where $|A| \geq |B|$. We will consider lists of size s taken from a color set of size s^5. We will show that there is a function $f(s)$ such that if $\delta(H) \geq f(s)$, then there is an assignment of lists to vertices and labels to edges such that there is no proper adaptable coloring from these lists. This is sufficient to show that $ch_a(H) > s$. This clearly is sufficient to prove Theorem 2.1, as we can let $g = f^{-1}$.

We start with a few helpful definitions. An assignment of lists to A (resp. B) is called an *A-set* (resp. *B-set*). Given a B-set, we say that $a \in A$ is *supersurrounded* (inspired by "surrounded" from [1]) if every possible list of s elements from $[s^5]$ appears in more than s^3 lists on vertices in $N(a)$ (the neighborhood of a). Furthermore, we call the B-set *bad* if at least half of the vertices in A are supersurrounded.

Theorem 2.1 follows directly from the following two lemmas.

Lemma 2.2. *If $\delta(H) \geq d = 36s^5\binom{s^5}{s}$, then there is a bad B-set of lists.*

Lemma 2.3. *There is an s_0 such that for any bad B-set $\mathcal{B}$, if $s \geq s_0$, there is an assignment of colors to the edges of H and an A-set $\mathcal{A}$ such that H does not have an acceptable coloring.*

Proof of Theorem 2.1. Let g be the inverse of the function $f(s) = 36s^5\binom{s^5}{s}$. We choose a bad B-set $\mathcal{B}$ according to Lemma 2.2. We choose an A-set $\mathcal{A}$ and an edge coloring according to Lemma 2.3 such that there is no acceptable coloring from the assigned lists. $\square$

Proof of Lemma 2.2. Uniformly at random assign lists to each of the vertices in B.

Let $a \in A$ be an arbitrary vertex and let Y be the number of lists which do not appear more than s^3 times in a's neighborhood. We will show that the probability that a is not supersurrounded, i.e. that $Y \geq 1$, is less than 1/2.

To make this computation it will be helpful to consider a single list. Let $S \subseteq [s^5]$ an arbitrary list of size s and let X be the number of neighbors of a whose assigned list is S.

Since the lists are assigned uniformly at random, for each neighbor b of a, the probability that b is assigned S is $1/\binom{s^5}{s}$. Therefore:

$$\mathbb{E}(X) = \frac{|N(a)|}{\binom{s^5}{s}} \geq \frac{d}{\binom{s^5}{s}} = 36s^5$$

The Chernoff bound yields the following.

$$\begin{aligned}\Pr\left(X \leq s^3\right) \leq \Pr\left(X \leq \frac{\mathbb{E}(X)}{2}\right) &\leq \Pr\left(|X - \mathbb{E}(X)| > \frac{\mathbb{E}(X)}{2}\right) \\ &< 2e^{-[\mathbb{E}(X)/2]^2/[3\mathbb{E}(X)]} \\ &= 2e^{-\mathbb{E}(X)/12} \leq 2e^{-36s^5/12} = 2e^{-3s^5}\end{aligned}$$

Now we can bound the expected value of Y using the linearity of expectation.

$$\mathbb{E}(Y) = \binom{s^5}{s} \Pr\left(X \leq s^3\right) < \binom{s^5}{s} 2e^{-3s^5} \leq 2e^{s^5}e^{-3s^5} < \frac{1}{2}\text{ , for every } s \geq 1.$$

Markov's Inequality yields that the probability that a is not supersurrounded is:

$$\Pr(Y \geq 1) \leq E(Y) < \frac{1}{2}$$

Now let Z be the number of vertices in A which are supersurrounded. By the linearity of expectation, $\mathbb{E}(Z) > \frac{1}{2}|A|$. Thus the probability that $Z \geq \frac{1}{2}|A|$ is positive, and therefore there is a bad B-set. $\square$

Proof of Lemma 2.3. Assume that $\mathcal{B}$ is a bad B-set.

Step 1: For each edge $e = ab$ where $a \in A$ and $b \in B$, assign to e a color uniformly at random from $L(b)$.

Consider any $a \in A$ that is supersurrounded. Fix a coloring of B from the lists of $\mathcal{B}$.

We will say that a color c is *available* for a if there is no neighbor b of a such that ab is labeled c and b is colored c. A coloring of B is *extendable* to A if every vertex in A has at least one available color in its list. Note that G is colorable if and only if at least one coloring of B is extendable to A.

First we note that all but at most $s-1$ colors appear more than s^2 times on vertices in the neighborhood of a. We can see this by assuming that $c_1, \ldots, c_s$ all appear at most s^2 times in $N(a)$. So the list $\{c_1, \ldots, c_s\}$ can only appear in $N(a)$ at most $s \cdot s^2 = s^3$ times. However, as a is supersurrounded, the list appears more than s^3 times and thus we have a contradiction.

Let c be a color that appears more than s^2 times in $N(a)$. The probability that a color c is available for a is the probability that for every neighbor b of a such that b is colored c, the edge $e = ab$ is not labeled c. Note that since we are choosing the color for e from b's list of colors, the probability that e is colored the same as b is $1/s$. Therefore:

$$\Pr(c \text{ is available}) < \left(1 - \frac{1}{s}\right)^{s^2} < e^{-s}.$$

Define Z to be the number of available colors beyond the $s-1$ colors which may appear s^2 or fewer times,

$$\mathbb{E}(Z) < s^5 e^{-s}$$

Using Markov's Inequality:

$$\Pr(Z \geq 1) \leq E(Z) < s^5 e^{-s}.$$

Now, including the $s-1$ colors which may appear s^2 or fewer times, we can with high probability bound the number of available colors as follows.

$$\Pr(\# \text{ available colors for } a \geq s) < s^5 e^{-s}. \tag{1}$$

Step 2: For each vertex $a \in A$, uniformly at random choose one of the $\binom{s^5}{s}$ possible lists.

Now, assuming that a is a vertex with fewer than s available colors, we can bound the probability that the list chosen for a has an available color. Since there are at most $s-1$ colors available for a, the probability that a random color c is available to a is at most $(s-1)/s^5$.

$$\Pr(\text{list chosen for } a \text{ contains an available color}) \leq s \cdot \frac{s-1}{s^5} < \frac{1}{s^3}$$

Therefore, by (1), the probability that a has s or more available colors or the list chosen for a has an available color is less than $s^5 e^{-s} + s^{-3}$. For sufficiently large s, this is less than $1/s^2$.

Since $\mathcal{B}$ is a bad B-set, there are at least $\frac{1}{2}|A|$ supersurrounded vertices. Thus, remembering that $|A| \geq |B|$, we can bound the probability that every supersurrounded vertex has an available color in its list as follows.

$$\Pr\left(\begin{array}{c}\text{every supersurrounded vertex}\\ \text{has an available color in its list}\end{array}\right) < \left(\frac{1}{s^2}\right)^{\frac{1}{2}|A|} = s^{-|A|} \leq s^{-|B|}$$

Let W be the number of colorings of B which are extendable to A. Given a B-set $\mathcal{B}$, there are $s^{|B|}$ possible ways of choosing colors for the vertices in B. Thus we can bound the expected value of W as follows.

$$\mathbb{E}(W) < s^{|B|} \cdot s^{-|B|} = 1$$

Since the expected value is less than 1, there must be a choice of an A-set and edge colorings such that no coloring of B can be extended to a coloring of A. $\square$

[1] N. Alon, Restricted colorings of graphs, *Surveys in Combinatorics, Proc. 14th British Combinatorial Conference, London Mathematical Society Lecture Notes Series* 187, edited by K. Walker, Cambridge University Press, 1993, pp. 1-33.

[2] A. Archer, On the upper chromatic numbers of the reals, *Discrete Math*, 214:65-75, 2000.

[3] H. Chernoff, A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Annuals of Mathematical Statistics*, 23:493-509, 1952.

[4] P. Erdős and A. Gyárfás, Split and balanced colorings of complete graphs, *Discrete Math*, 200 (1999), 79-86.

[5] J. Esperet, M. Montassier and X. Zhu, Adapted list colouring of planar graphs, submitted.

[6] T. Feder and P. Hell, Full constraint satisfaction problems, *SIAM J. Computing*, 36 (2006), 230-246.

[7] T. Feder, P. Hell, S. Klein and R. Motwani, Complexity of list partitions, *SIAM J. Discrete Mathematics*, 16 (2003), 449-478.

[8] T. Feder, P. Hell and W. Xie, Matrix partitions with finitely many obstructions, *Electronic Notes in Discrete Math*, 28 (2007), 371-378.

[9] A. Guan and X. Zhu, Adaptive choosability of planar graphs with sparse short cycles, submitted.

[10] P. Hell, Z. Pan, T. Wong and X. Zhu, Adaptable colouring of graph products, preprint.

[11] P. Hell and X. Zhu, Adaptable chromatic number of graphs, *European J. Combinatorics*, 29(4) (2008), 912-921.

[12] A. Kostochka and X. Zhu, Adapted list coloring of graphs and hypergraphs, *SIAM J. Discrete Math*, 22 (2008), no. 1, 398–408.

[13] M. Molloy and B. Reed, *Graph colouring and the probabilistic method*, Springer-Verlag, Berlin, 2001.

[14] M. Molloy and G. Thron, An asymptotically tight bound on the adaptable chromatic number, *J. Graph Theory*, to appear.

[15] M. Montassier, A. Raspaud and X. Zhu, An upper bound on adaptable choosability of graphs, *European Journal of Combinatorics*, 30 (2009), 351-355.