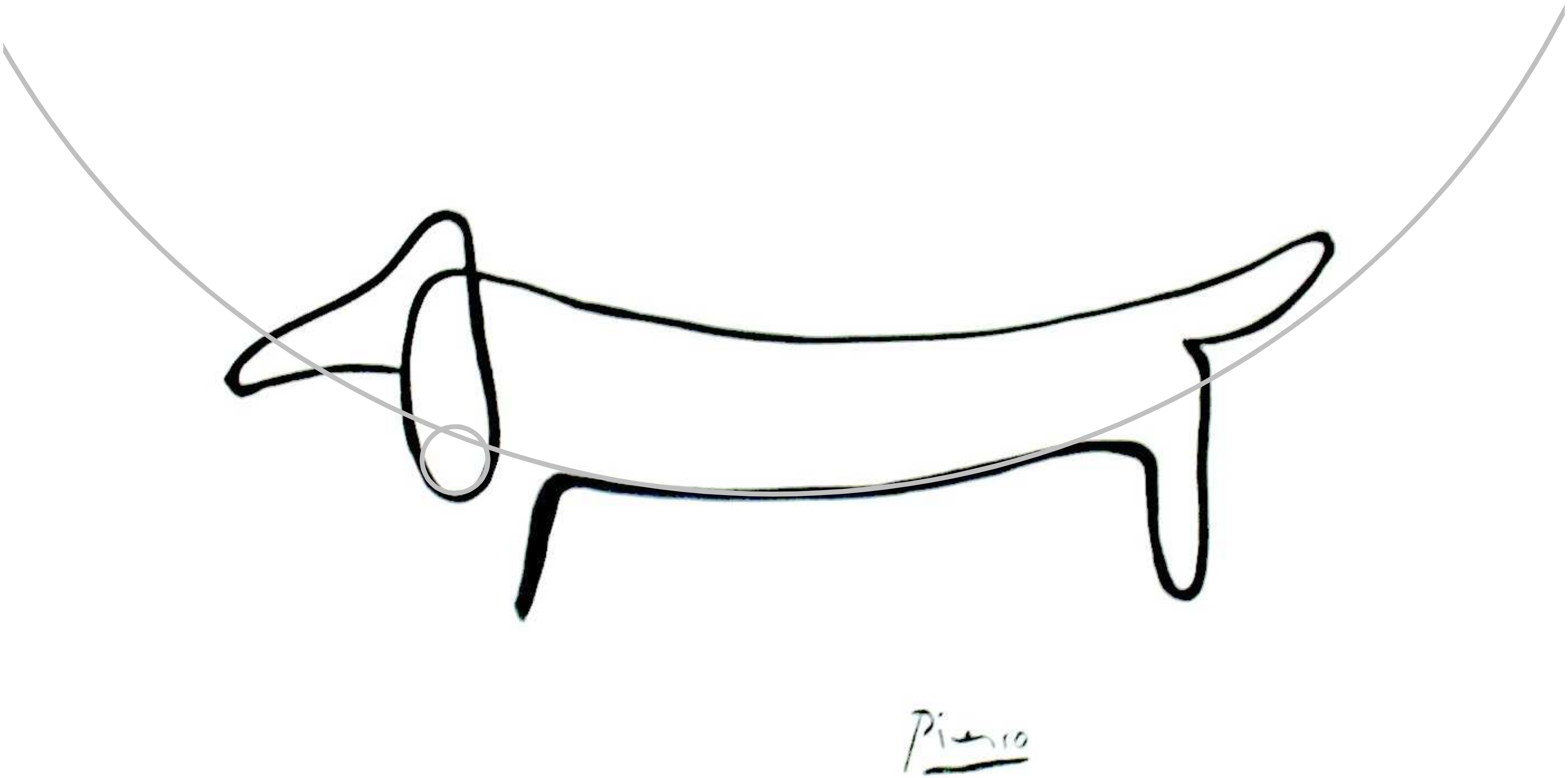
Local Analysis of 2D Curve Patches



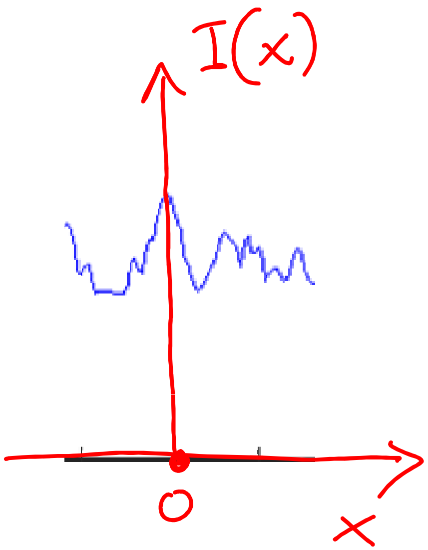
Topic 4.2:

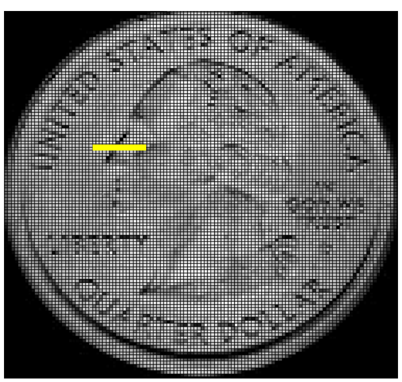
Local analysis of 2D curve patches

- Representing 2D image curves
- Estimating differential properties of 2D curves
    - Tangent & normal vectors
    - The arc-length parameterization of a 2D curve
    - The curvature of a 2D curve

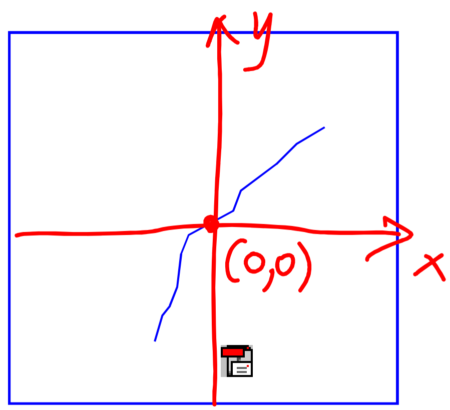
Local Analysis of Image Patches: Outline

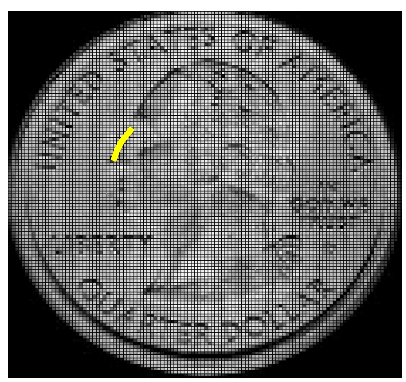
As graph in 2D



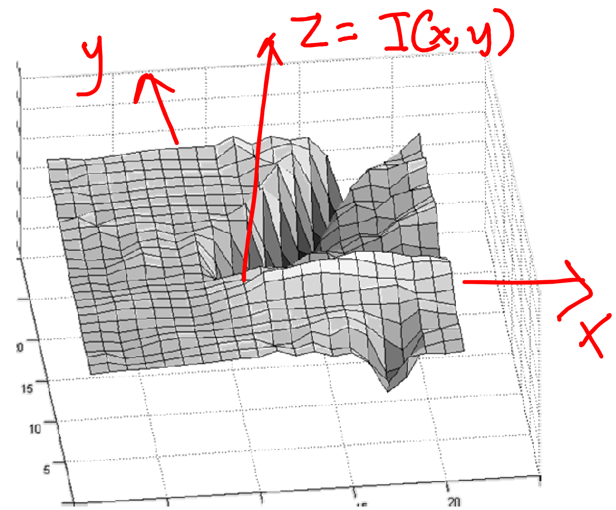


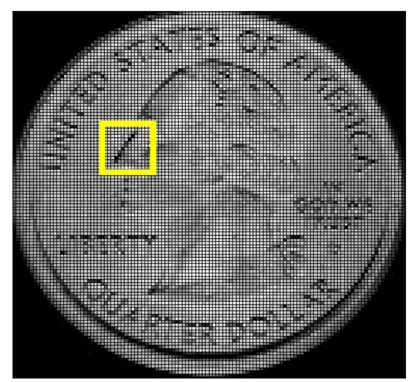
As curve in 2D





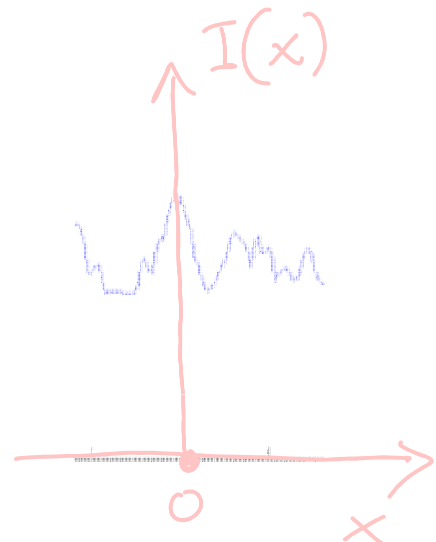
As surface in 3D

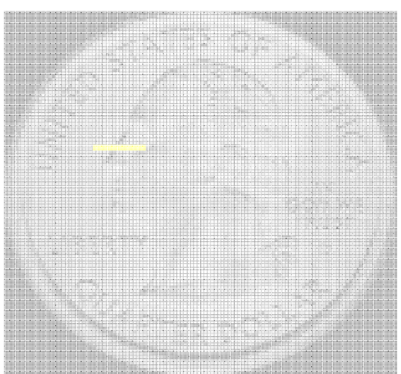




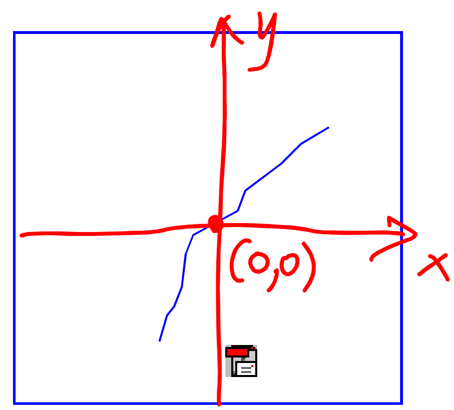
Local Analysis of Image Patches: Outline

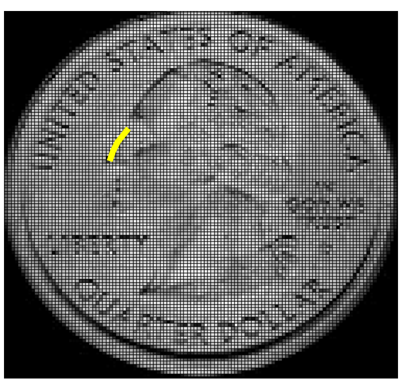
As graph in 2D



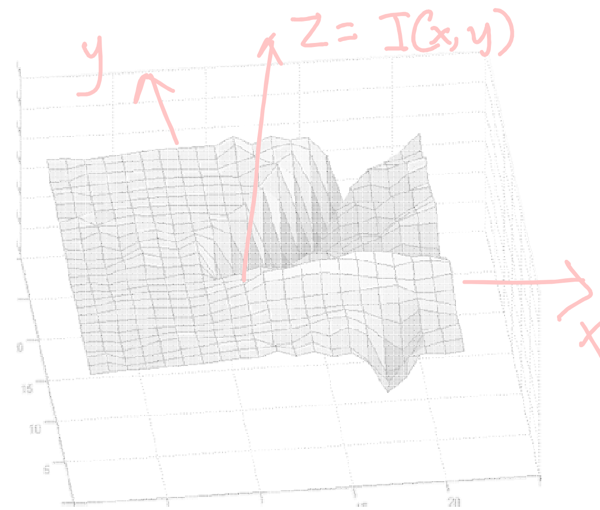


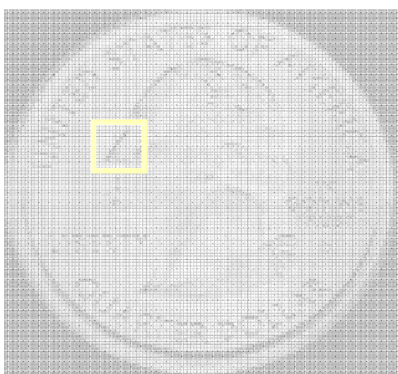
As curve in 2D



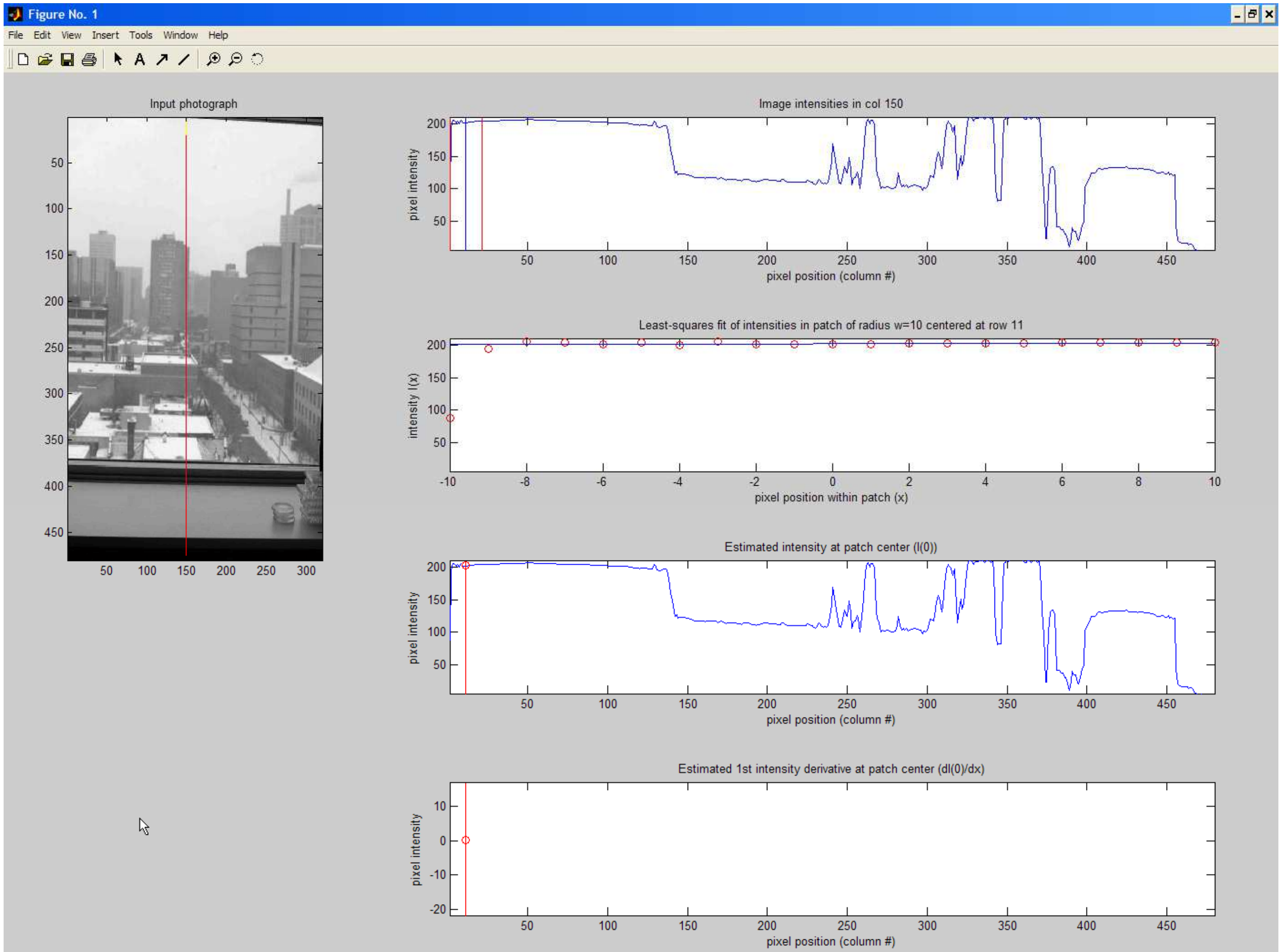


As surface in 3D



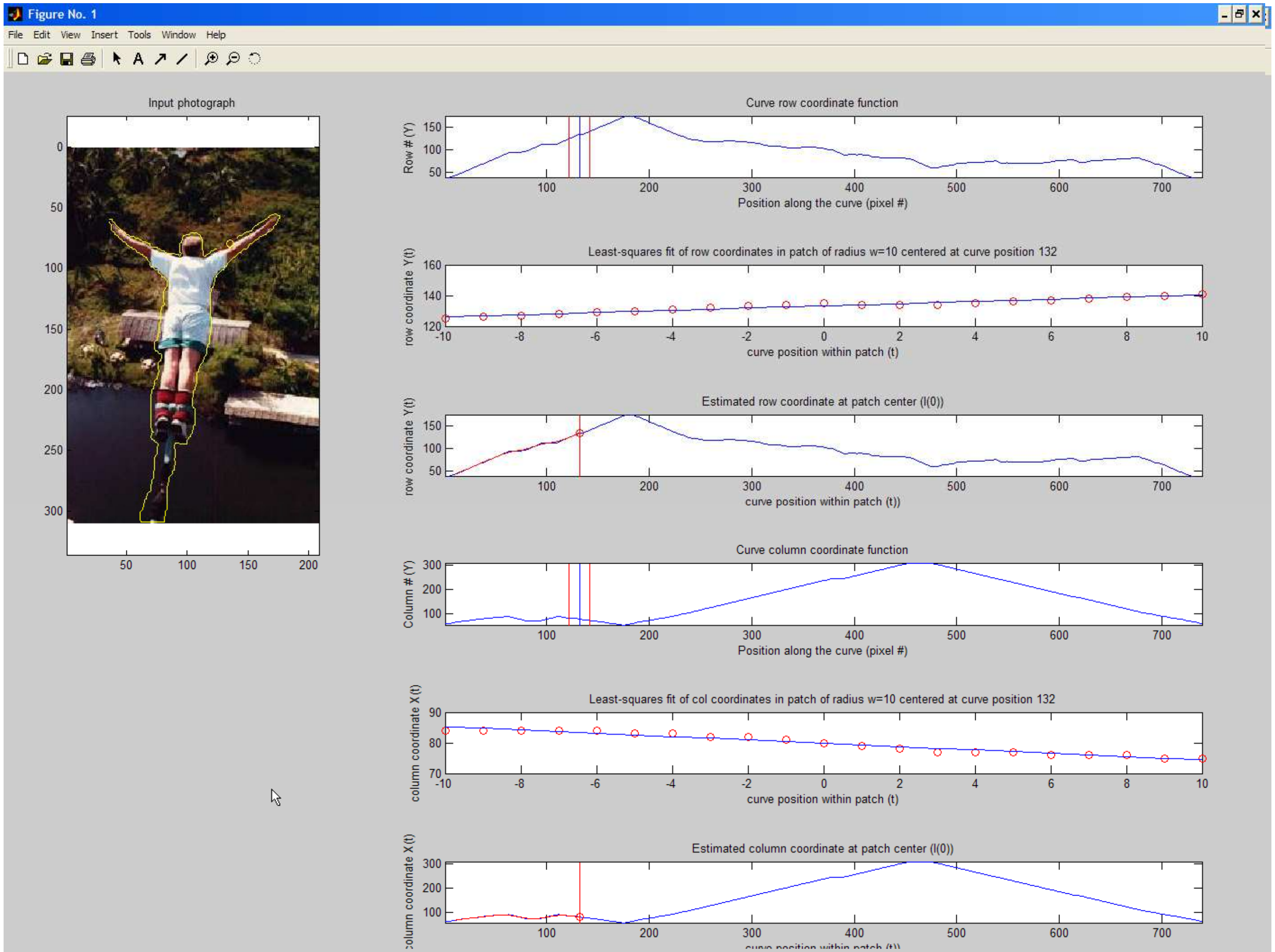


Estimating Intensities & their Derivatives



Don’t go, or you’ll miss out!

Estimating Intensities & their Derivatives



Don't go, or you'll miss out!

Representing & Analysing 2D Curves, why?

- Useful representation for:
  - Object boundaries
  - Isophote regions (groups of pixels with the same intensity)

Representing & Analysing 2D Curves, how?

Math is our friend:

- Provides an unambiguous representation
- Enables computation of useful properties

2D Image Curves: Definition

A parametric 2D curve is a continuous mapping

$$\gamma : (a,b) \to R^2$$

value of parameter at beginning

value of parameter at end

where

$$t \to \underbrace{(x(t), y(t))}$$

curve parameter (indicates position along the curve)

point along the curve at position t

2D Image Curves: Definition

Example: a boundary curve

t = pixel # along the boundary

x(t) = x coordinate of the t^{th} pixel

y(t) = y coordinate of the t^{th} pixel

2D Image Curves: Definition

To fully describe a curve we need the two functions x(t) and y(t), called the Coordinate Functions.

2D Image Curves: Definition

A closed 2D curve is a continuous mapping

$$\gamma : (a,b) \to R^2$$

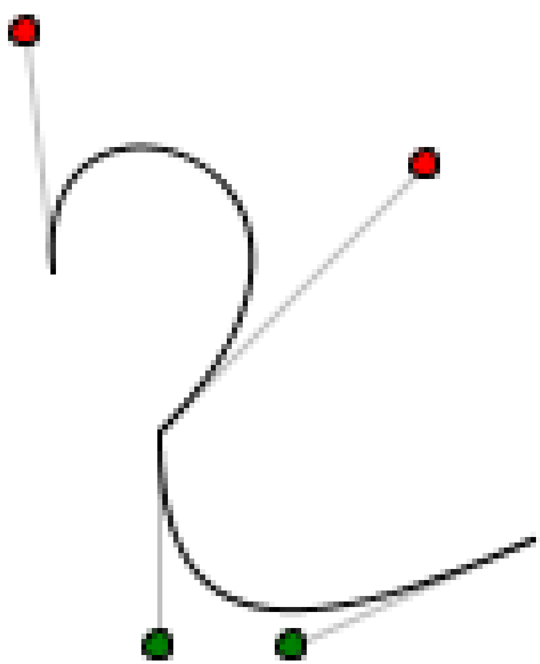
value of parameter at beginning

value of parameter at end

where $t \to (x(t), y(t))$

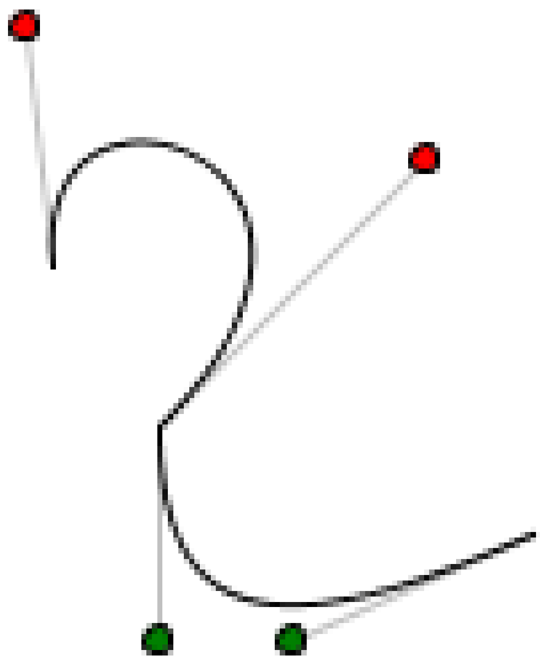
such that $(x(a), y(a)) = (x(b), y(b))$.

Smooth 2D Curves



A curve is smooth when...

Smooth 2D Curves



A curve is smooth when all the derivatives of the Coordinate Functions exist

$$\frac{d^n x}{dt}(t), \frac{d^n y}{dt}(t)$$

for all n, t

Derivatives of the Coordinate Functions

The 1st and 2nd derivatives of $x(t)$, $y(t)$ are extremely informative about the shape of a curve.

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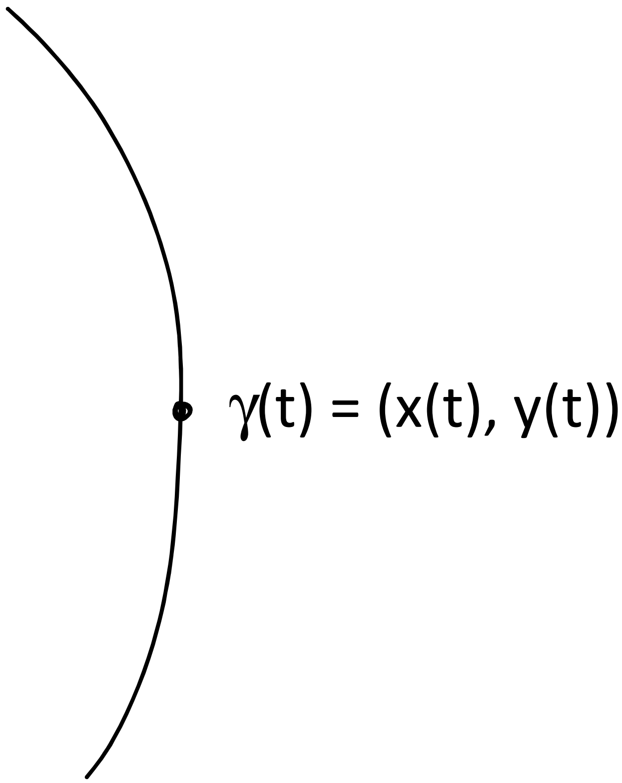
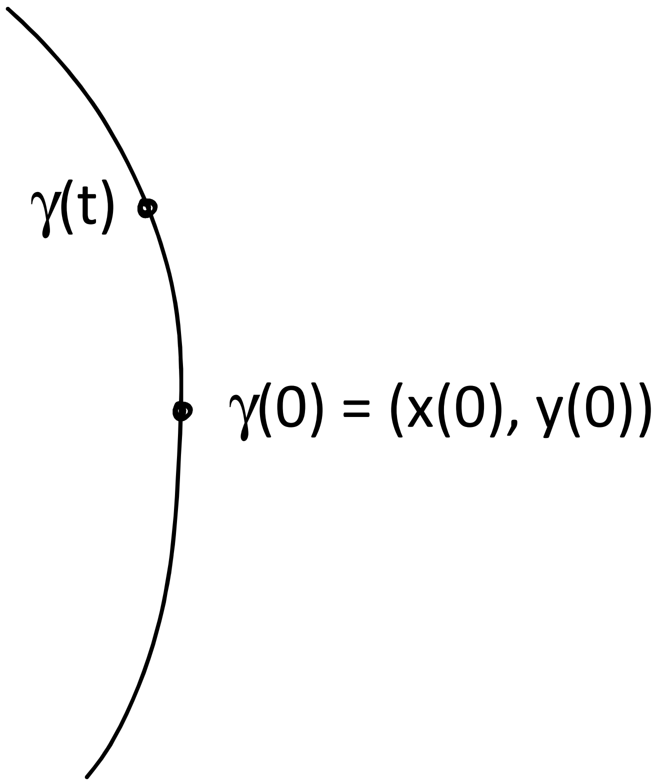
The Tangent Vector



Notation:

- $\gamma(t)$ maps a number (t) to a 2D point (x(t), y(t)).
- This type of function is called a vector-valued function.

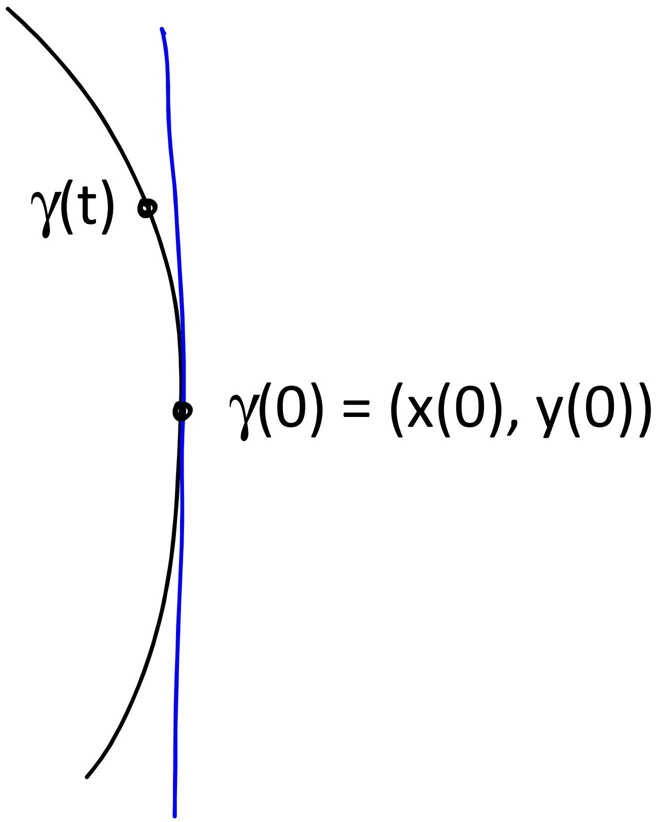
The Tangent Vector



Suppose we know $\gamma(0)$.

How can we approximate $\gamma(t)$?

The Tangent Vector

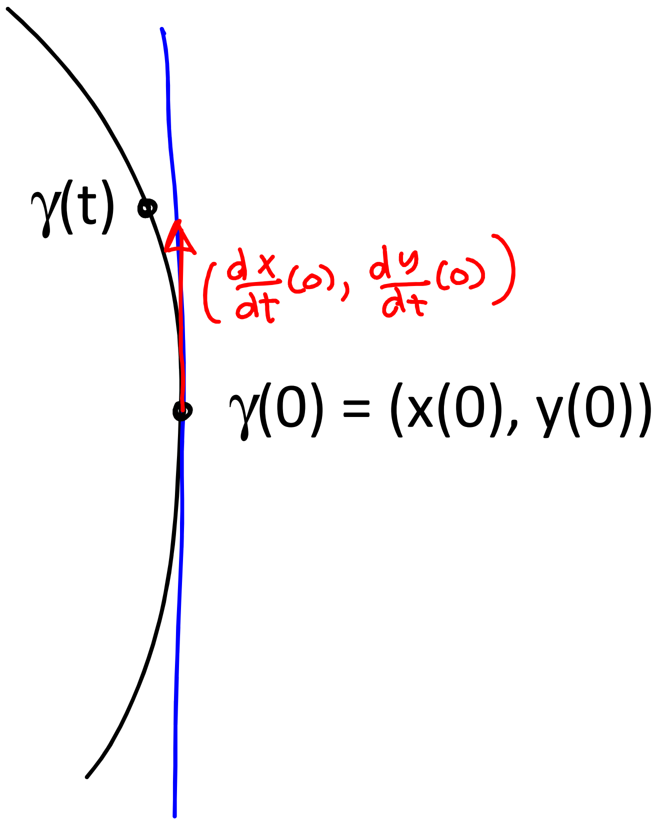


Suppose we know $\gamma(0)$.

How can we approximate $\gamma(t)$?

hint?

The Tangent Vector

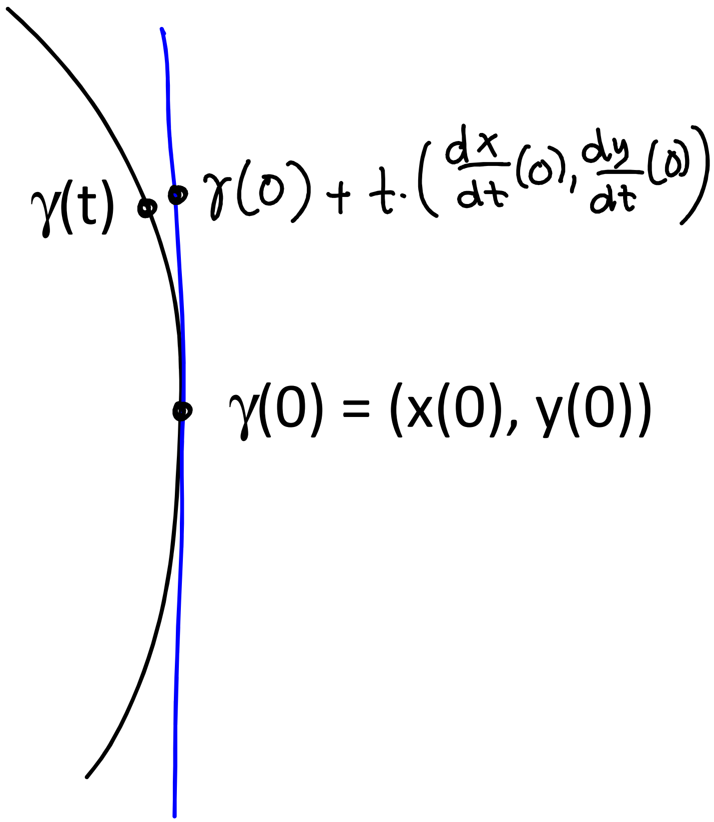


Suppose we know $\gamma(0)$.

How can we approximate $\gamma(t)$?

Using the derivative (tangent)!

The Tangent Vector



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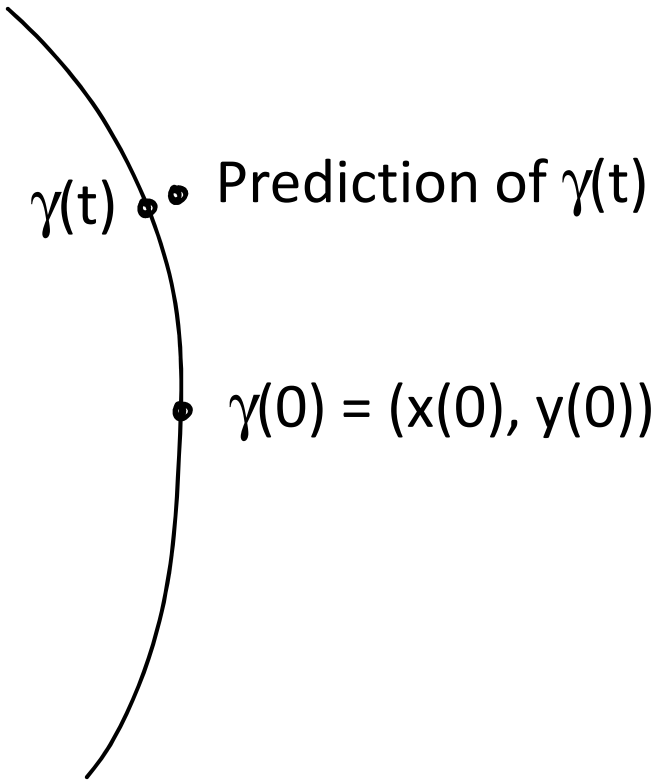
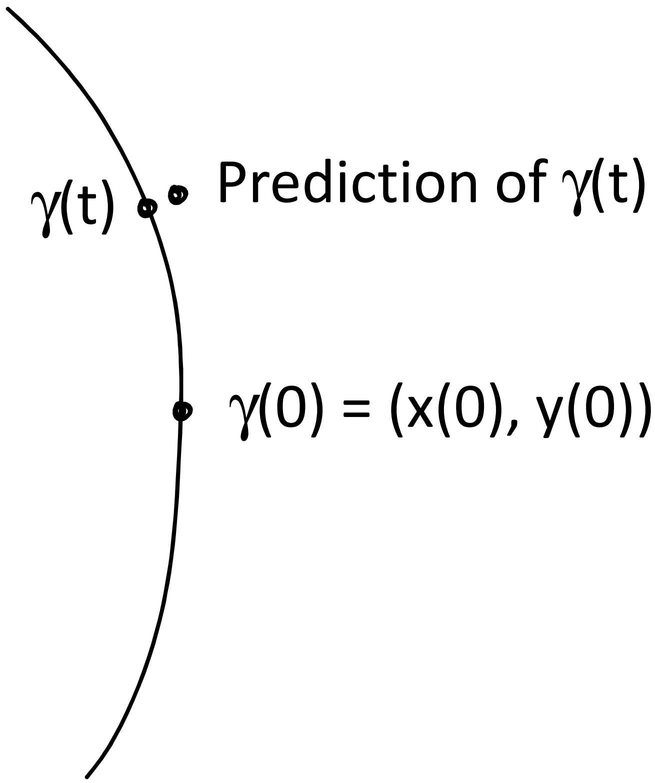


Good! But not great.

Can we do any better?

If so, how?

The Tangent Vector

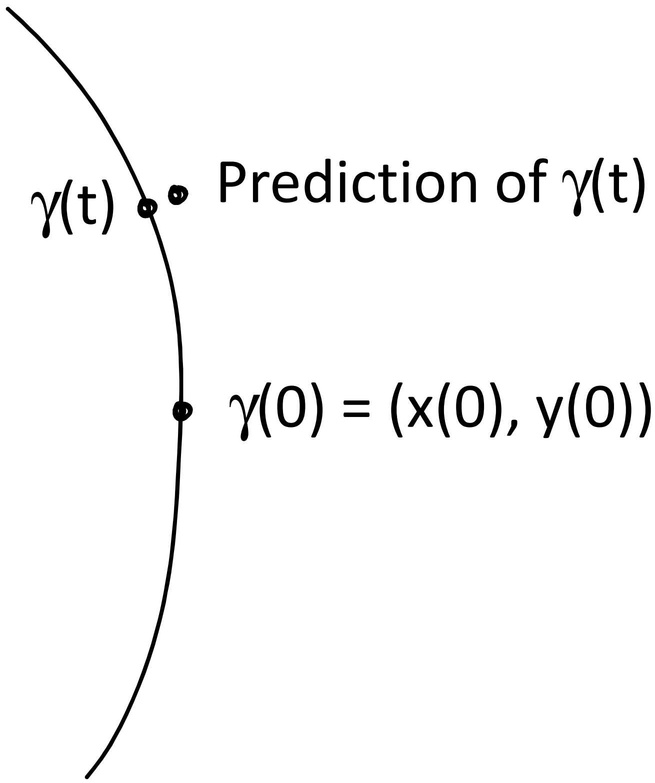


Good! But not great.

Add more information about the curve, like the 2^{nd}, 3^{rd},… or the n^{th} derivative!

Familiar?

The Tangent Vector



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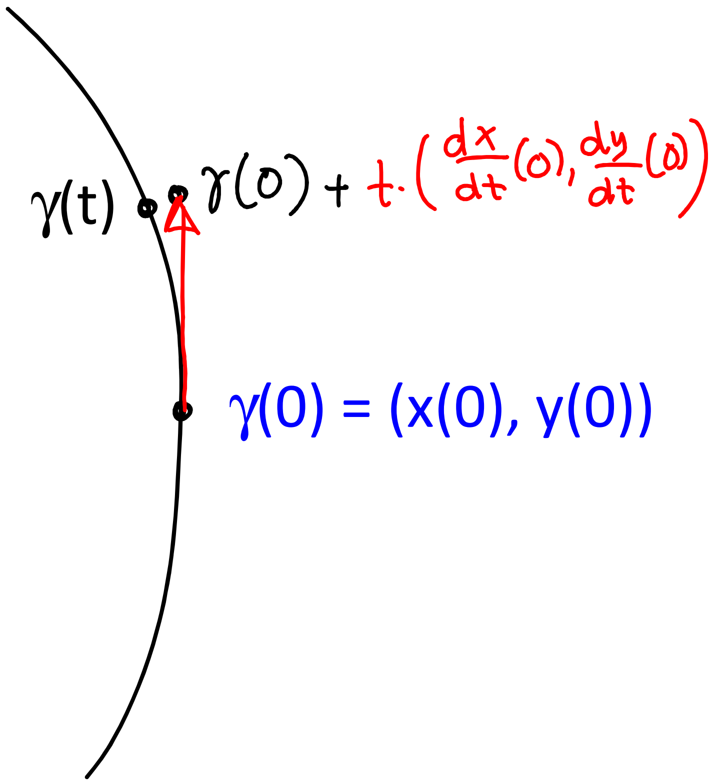
This is a Taylor-Series approximation

The Tangent Vector



Formally: the 1st order Taylor-Series approximation to $\gamma(t)$ near $\gamma(0)$ is:

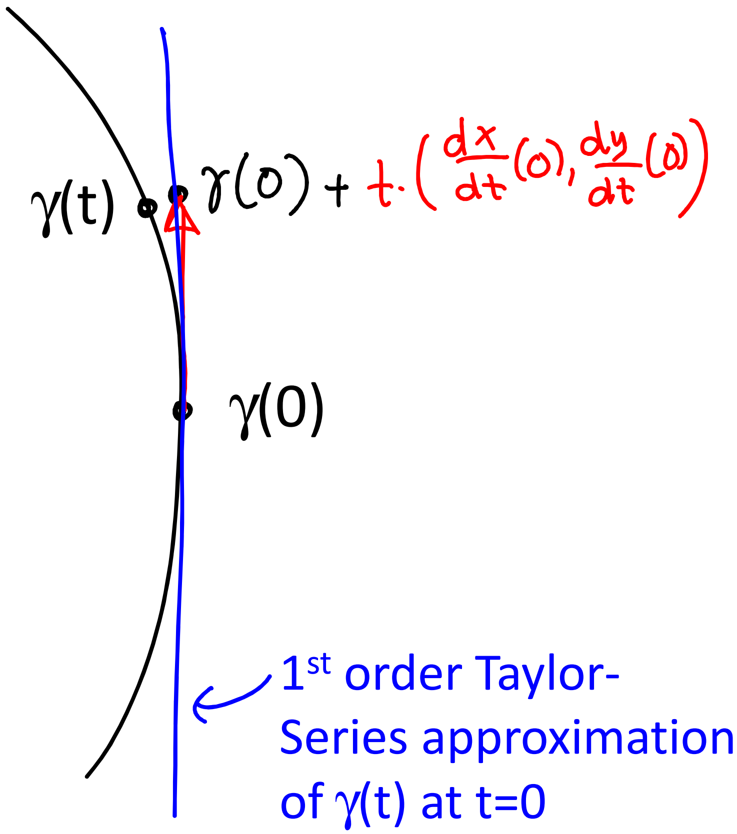
$$\gamma(\Delta t) = \gamma(0) + \Delta t \cdot \frac{d\gamma}{dt}(0)$$, so

$$\gamma(t) = (x(t), y(t))$$

$$\simeq \left(x(0) + t \cdot \frac{dx}{dt}(0),\; y(0) + t \cdot \frac{dy}{dt}(0) \right)$$

$$= \underbrace{(x(0), y(0))}_{\text{point}} + \underbrace{t \cdot \left(\frac{dx}{dt}(0), \frac{dy}{dt}(0) \right)}_{\text{tangent vector}}$$

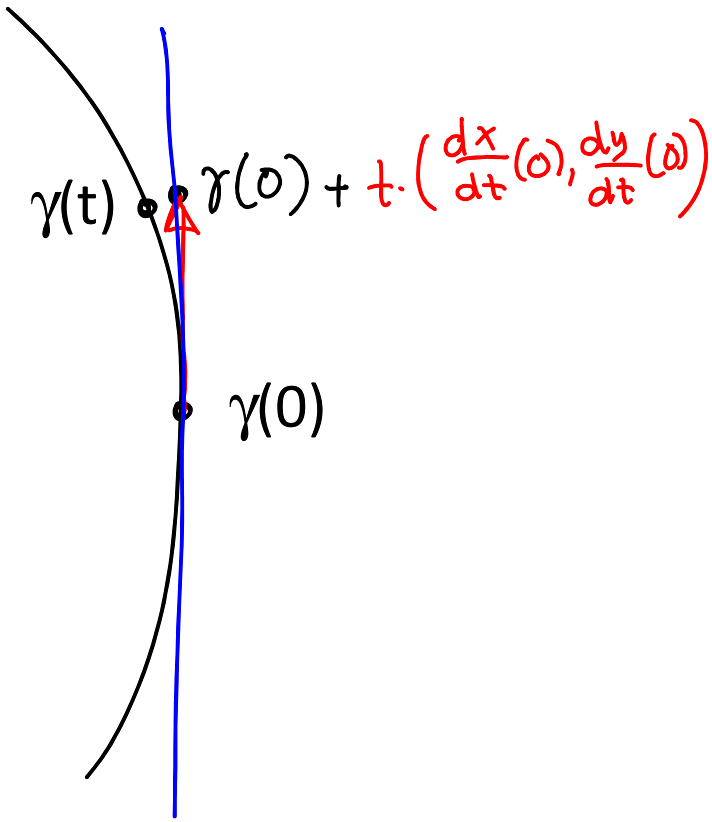
The Tangent Vector



Definition.
The tangent vector at $\gamma(t)$ is equal to the first derivative of the function, at that point. In this case:

$$\frac{d\gamma}{dt}(t) = \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

The Tangent Vector



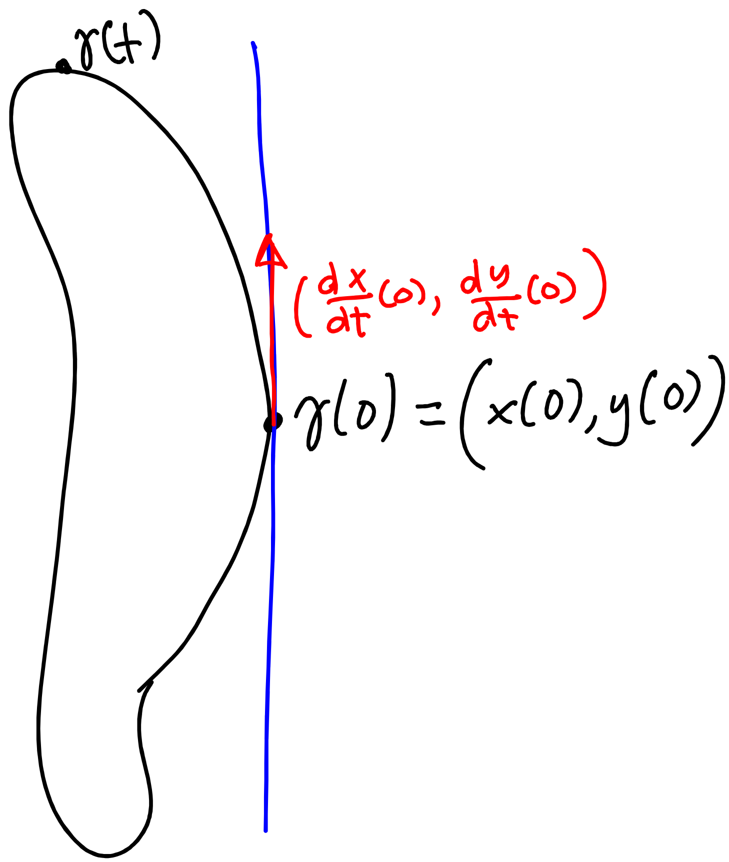
In general, the derivative of a vector valued function is the derivative of the n coordinate functions, so if

$$f(t) = \left(f_1(t), \ldots, f_n(t) \right)$$

The derivative of f at (t) is:

$$\frac{df}{dt}(t) = \left(\frac{df_1}{dt}(t), \ldots, \frac{df_n}{dt}(t) \right)$$

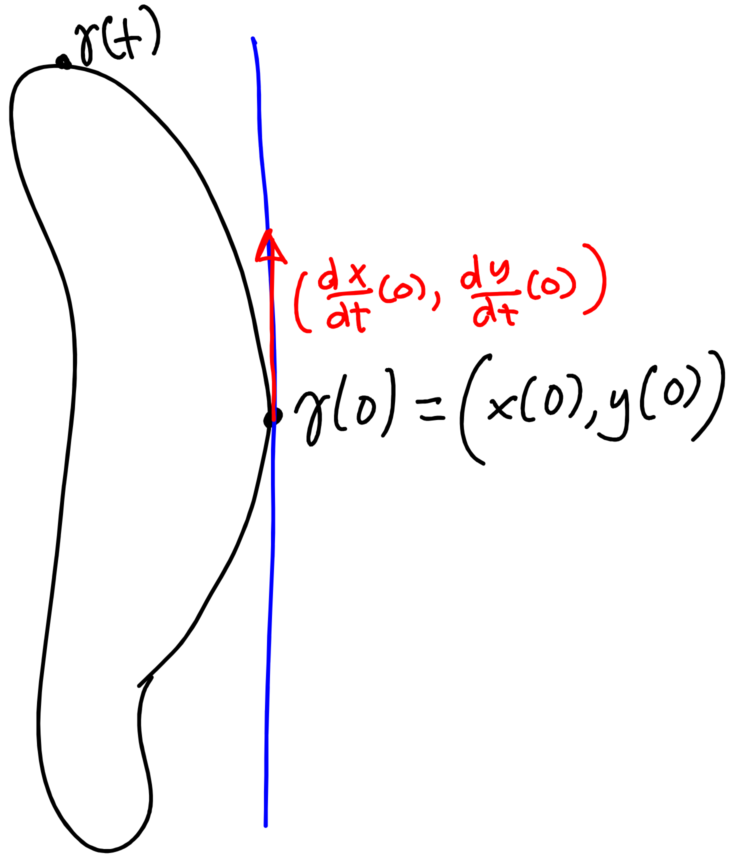
Effect of Curve Parameter on the Tangent



We can parameterize a curve γ in (infinitely) many different ways, for instance:

1. Make t the number of pixels between $\gamma(0)$ and $\gamma(t)$
2. Make t the actual length of the curve between $\gamma(0)$ and $\gamma(t)$, in meters (or inches, or light-years).

Effect of Curve Parameter on the Tangent

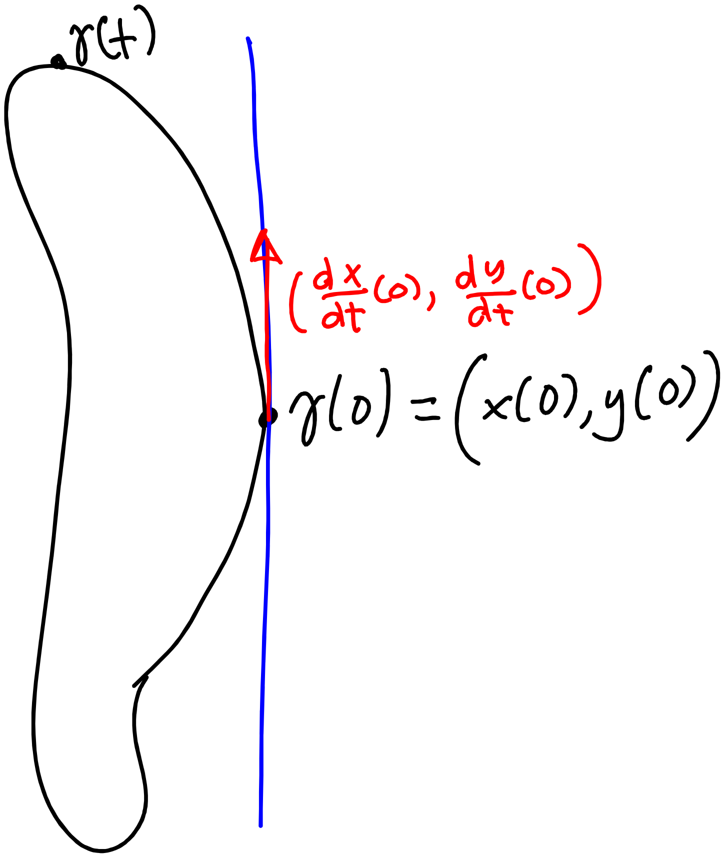


We can parameterize a curve γ in (infinitely) many different ways, for instance:

1. Make t the number of pixels between γ(0) and γ(t)
2. Make t the actual length of the curve between γ(0) and γ(t), in meters (or inches, or light-years).

But the key property is that the direction of the tangent remains unchanged, regardless of the scale of the parameter.

Effect of Curve Parameter on the Tangent



The direction of the tangent remains unchanged, regardless of the scale of the parameter.

Really?

Can we prove it?

Effect of Curve Parameter on the Tangent

Proof:

Let's parameterize the curve γ in two ways:

1. Take t = the number of pixels between $\gamma(0)$ and $\gamma(t)$
2. Take s = f(t) as the parameter, where f(t) is simply any differentiable function.

Effect of Curve Parameter on the Tangent

Proof:

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In 1, we know the derivative of γ is simply $\frac{d\gamma}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$

Effect of Curve Parameter on the Tangent

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In 1, we know the derivative of γ is simply $\frac{d\gamma}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

In 2, the chain rule tells us that if s=f(t) and $\gamma(s)$ then:

$$\frac{d\gamma}{dt} = \left(\frac{dx}{ds}\cdot\frac{df}{dt}, \frac{dy}{ds}\cdot\frac{df}{dt}\right) = \frac{df}{dt}\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$

which correspond to

Effect of Curve Parameter on the Tangent

Proof:

Let's parameterize the curve γ in two ways:

1. Take t = the number of pixels between $\gamma(0)$ and $\gamma(t)$
2. Take s = f(t) as the parameter, where f(t) is simply any differentiable function.

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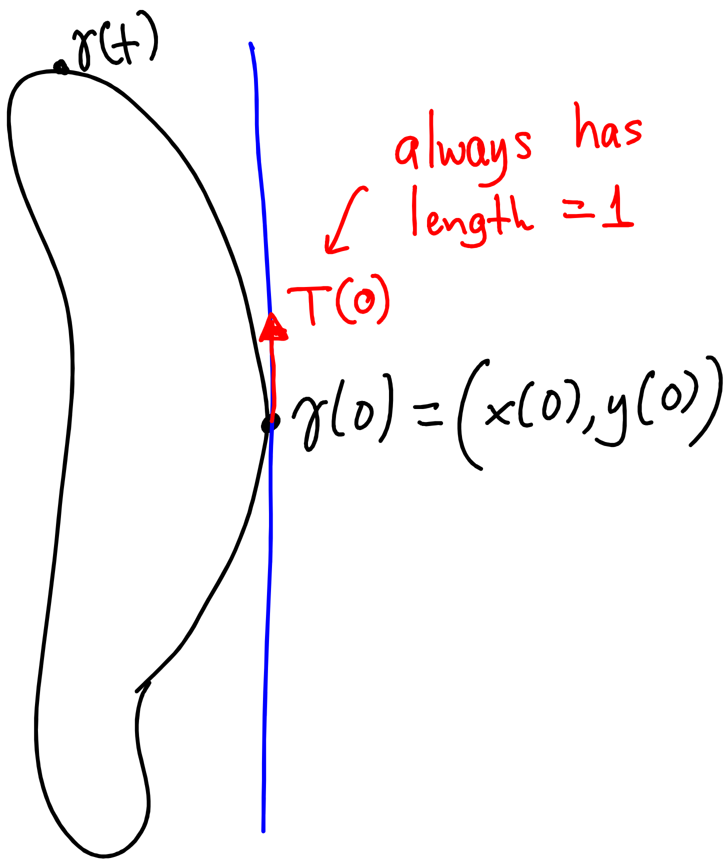
In 2, the chain rule tells us that if s=f(t) and $\gamma(s)$ then:

$$\frac{d\gamma}{dt} = \left(\frac{dx}{ds} \cdot \frac{df}{dt}, \frac{dy}{ds} \cdot \frac{df}{dt} \right) = \underbrace{\frac{df}{dt}}_{\text{multiplicative scalar}} \underbrace{\left(\frac{dx}{ds}, \frac{dy}{ds} \right)}_{\frac{d\gamma}{ds}}$$

The Unit Tangent Vector

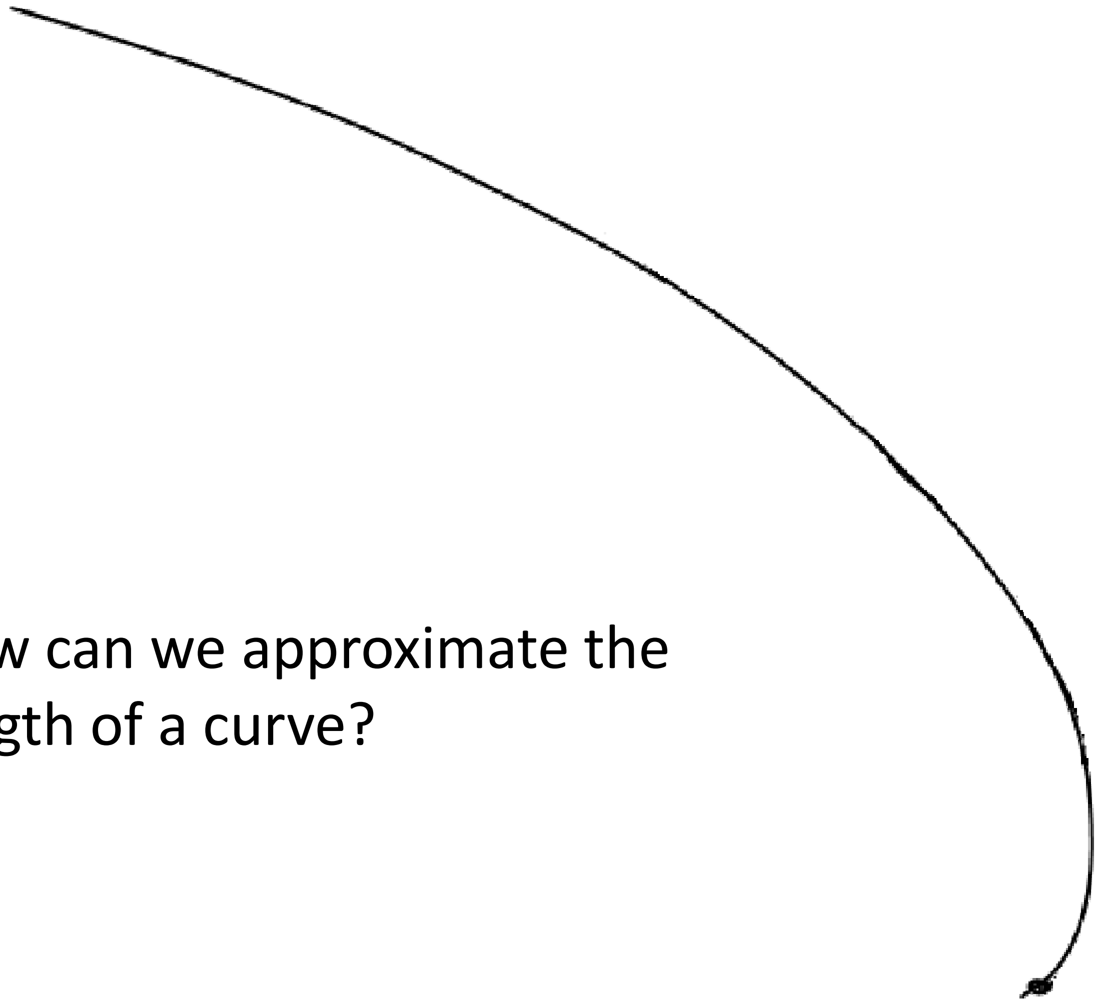


Definition. The Unit Tangent is:

$$T(t) = \frac{d\gamma}{dt}(t) \cdot \frac{1}{\left\| \frac{d\gamma}{dt}(t) \right\|_2}$$

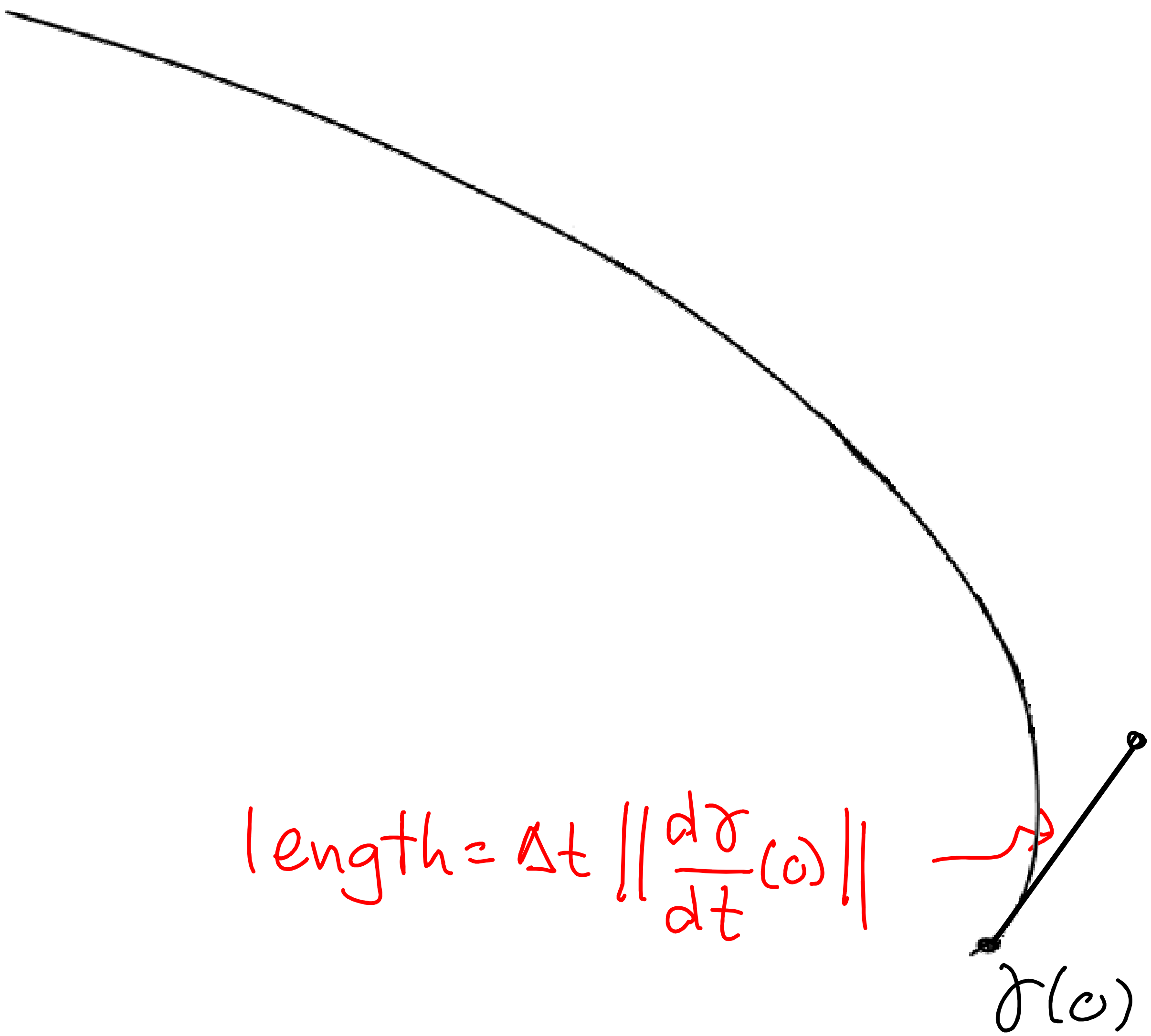
The unit tangent vector does not depend on the choice of the parameter t

The Arc-Length of a Curve



How can we approximate the length of a curve?

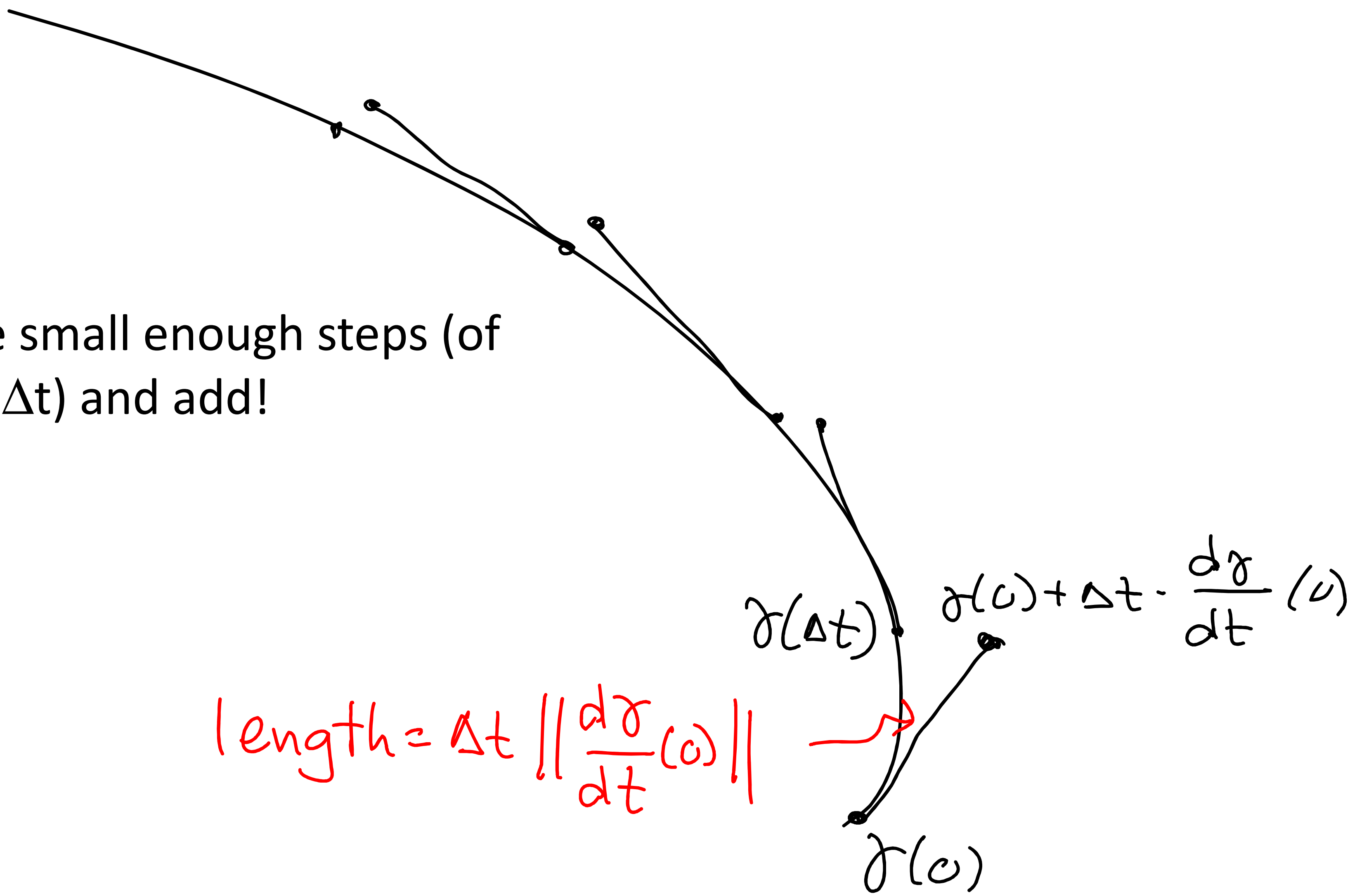
The Arc-Length of a Curve



$$\text{length} = \Delta t \left\| \frac{d\gamma}{dt}(0) \right\|$$

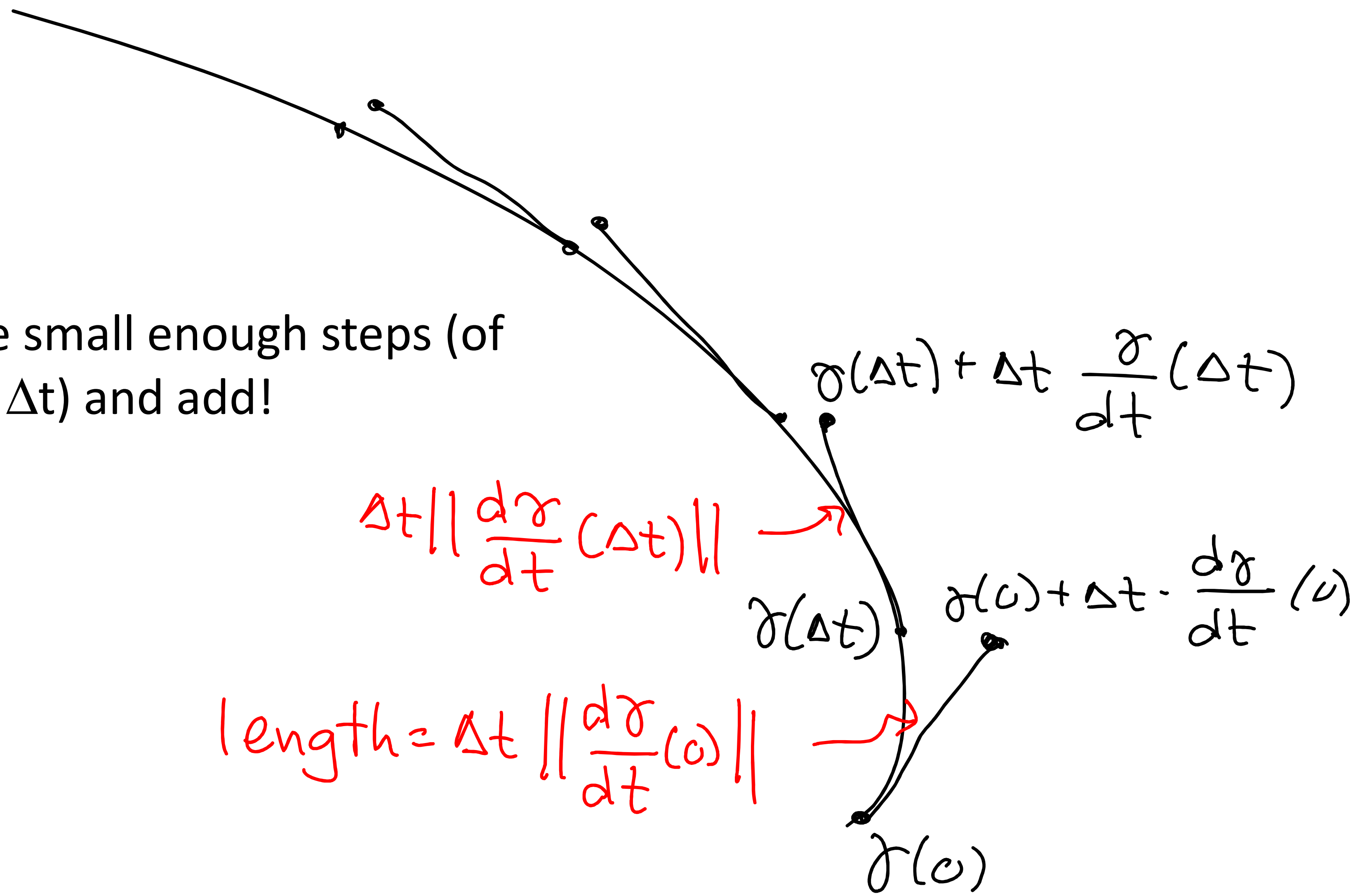
The Arc-Length of a Curve

Take small enough steps (of size Δt) and add!

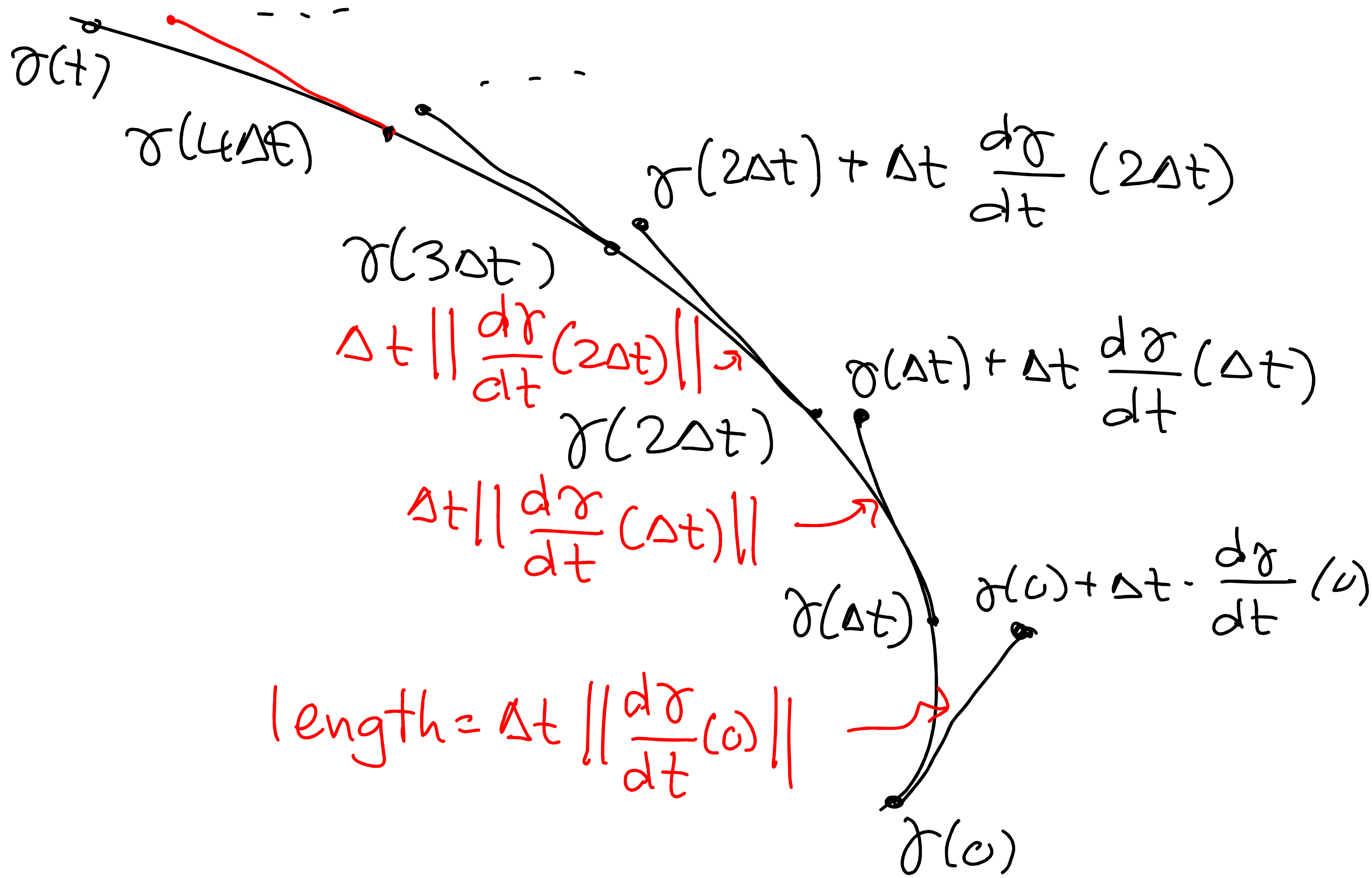


The Arc-Length of a Curve

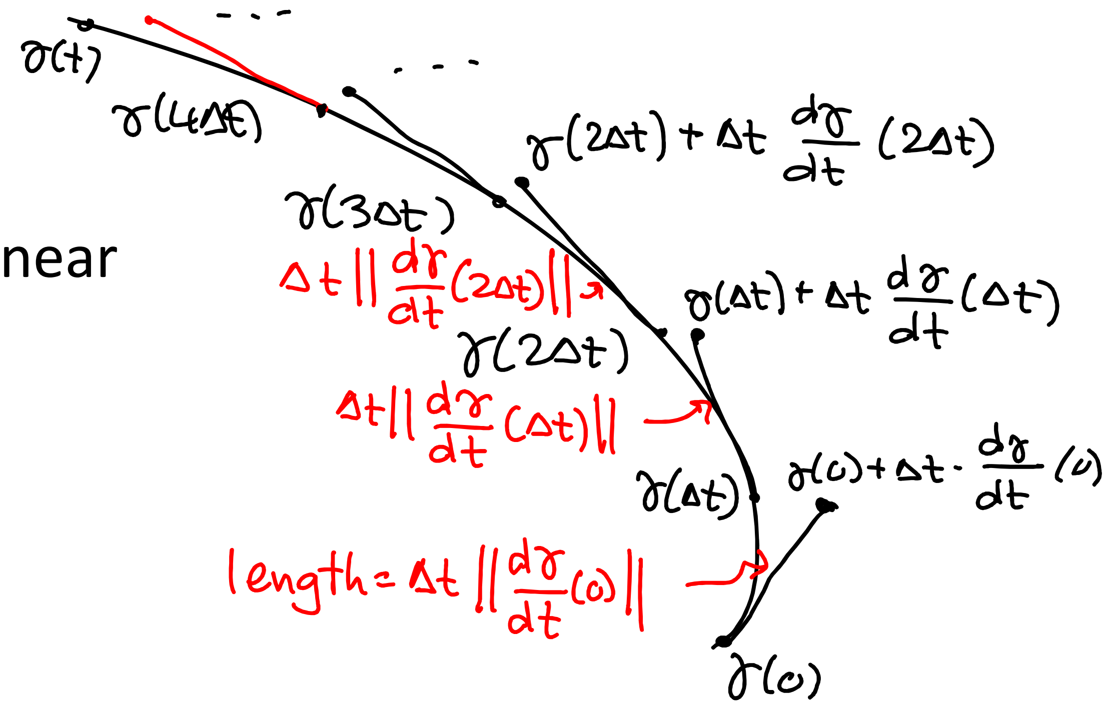
Take small enough steps (of size Δt) and add!



The Arc-Length of a Curve



The Arc-Length of a Curve



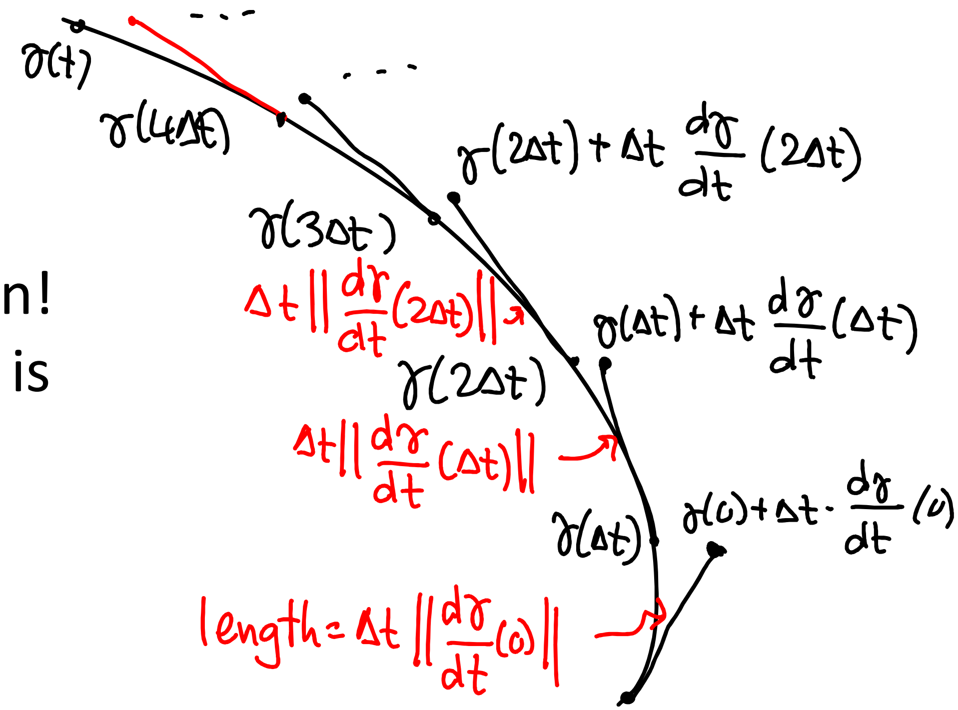
This is called a piece-wise-linear length approximation.

$$S(t) = \Delta t \left\| \frac{d\gamma}{dt}(0) \right\| + \Delta t \left\| \frac{d\gamma}{dt}(\Delta t) \right\| + \cdots$$

And what if we make the steps smaller and smaller?

$\Delta t \to 0$

The Arc-Length of a Curve



Then we get the following definition!
The arc-length s(t) of the curve γ(t) is given by:

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$$

The Arc-Length of a Curve

For example, lets think about the circle

What do we expect

The Arc-Length of a Curve

For example, lets think about the circle

Proportional to the radius and the number of pixels in the circle

The Arc-Length of a Curve

Example: The arc length of a circle with radius r, whose curve equation can be written as:

$$\gamma(t) = r\,(\cos(t), \sin(t))$$

then $$\frac{d\gamma}{dt}(t) = r\underbrace{(-\sin t, \cos t)}_{\text{unit vectors}}$$

so $$\left\|\frac{d\gamma}{dt}(t)\right\| = r$$

$$s(t) = \int_0^t \left\|\frac{d\gamma}{dt}(u)\right\| du$$

The Arc-Length of a Curve

Example: The arc length of a circle with radius r, whose curve equation can be written as:

$$\gamma(t) = r\,(\cos(t), \sin(t))$$

$$\left\| \frac{d\gamma}{dt}(t) \right\| = r$$

Now, substituting on the definition, we get:

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du = \int_0^t r\,du = rt$$

The Arc-Length of a Curve

Example: The arc length of a circle with radius r, whose curve equation can be written as:

$$\gamma(t) = r\ (\ \cos(t), \sin(t)\ )$$

$$\left\|\frac{d\gamma}{dt}(t)\right\| = r$$

Now, substituting on the definition, we get:

$$s(t) = \int_0^t \left\|\frac{d\gamma}{dt}(u)\right\| du = \int_0^t r\,du = rt$$

Proportional to the radius… yes!

Proportional to the number of pixels in the circle… yes!

The Arc-Length of a Curve

Now, can we parameterize the function $\gamma(t)$ using the arc-length function itself?

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$$

The Arc-Length of a Curve

Now, can we parameterize the function $\gamma(t)$ using the arc-length function itself?

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$$

Yes!

The Arc-Length of a Curve

Now, can we parameterize the function $\gamma(t)$ using the arc-length function itself?

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$$

Yes!

A parameterization $\gamma(s)$ where the curve parameter is the arc-length is (thoughtfully and originally) named the arc-length parameterization.

The Arc-Length of a Curve

Now, can we parameterize the function $\gamma(t)$ using the arc-length function itself?

$$s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$$

Yes!

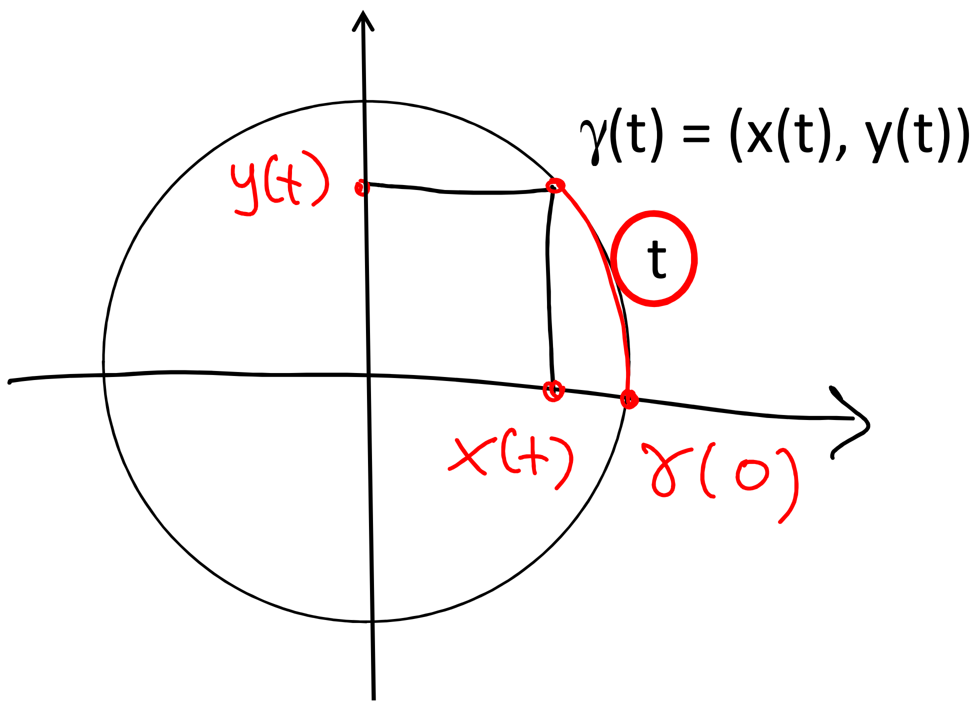
Lets use the circle again as an example. We know that the arc-length of a circle is s(t) = rt, or for short s = rt.

Which means that $t = \frac{s}{r}$, so the arc-length parameterization is: $r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$

The Arc-Length of a Curve

Arc-length parameterization of the circle.

Using $\gamma(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$ the following holds:



The Arc-Length of a Curve

Now, we know that the arc-length is $s(t) = \int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du$

We also know that an arc-length parameterization $\gamma(s)$ is one where the curve parameter is the arc-length

Knowing these two facts, a property we can derive is that $\gamma(s)$ is an arc-length parameterization of a curve if and only if

$$\left\| \frac{d\gamma}{ds} \right\| = 1$$

The Arc-Length of a Curve

$\gamma(s)$ is an arc-length parameterization of a curve if and only if

$$\left\| \frac{d\gamma}{ds} \right\| = 1$$

Proof:

$$\frac{d\gamma}{dt} = \frac{d\gamma}{ds} \cdot \frac{ds}{dt} \quad \text{(chain rule)}$$

$$\Leftrightarrow \frac{d\gamma}{dt} = \frac{d\gamma}{ds} \cdot \frac{d}{dt}\left(\int_0^t \left\| \frac{d\gamma}{dt}(u) \right\| du \right)$$

$$\Leftrightarrow \frac{d\gamma}{dt} = \frac{d\gamma}{ds} \cdot \left\| \frac{d\gamma}{dt} \right\|$$

$$\Leftrightarrow \left\| \frac{d\gamma}{dt} \right\| = \left\| \frac{d\gamma}{ds} \right\| \cdot \left\| \frac{d\gamma}{dt} \right\|$$

$$\Leftrightarrow \left\| \frac{d\gamma}{ds} \right\| = 1$$

The Arc-Length of a Curve

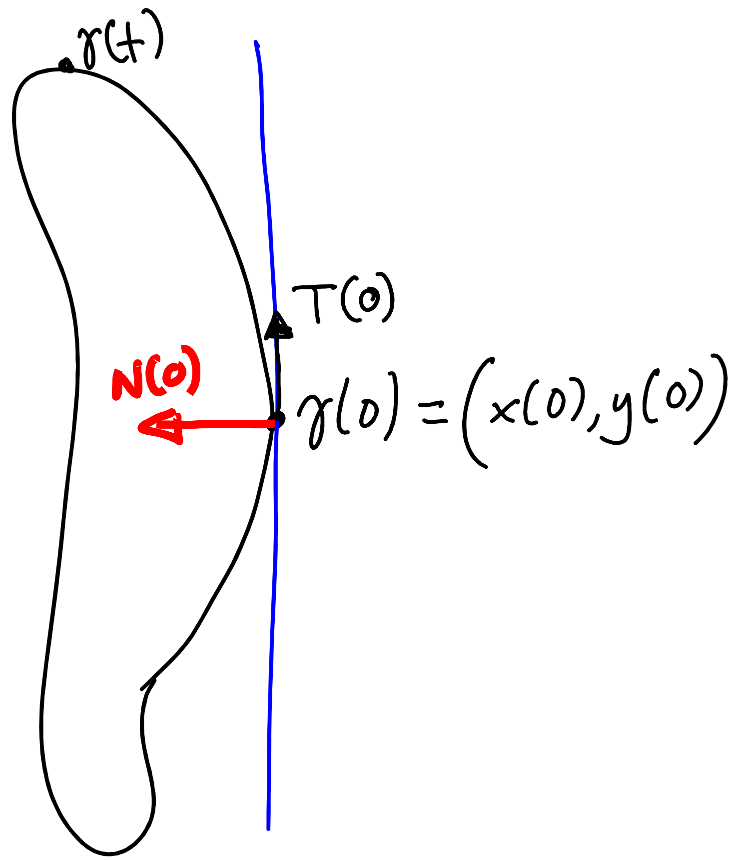
$\gamma(s)$ is an arc-length parameterization of a curve if and only if

$$\left\| \frac{d\gamma}{ds} \right\| = 1$$

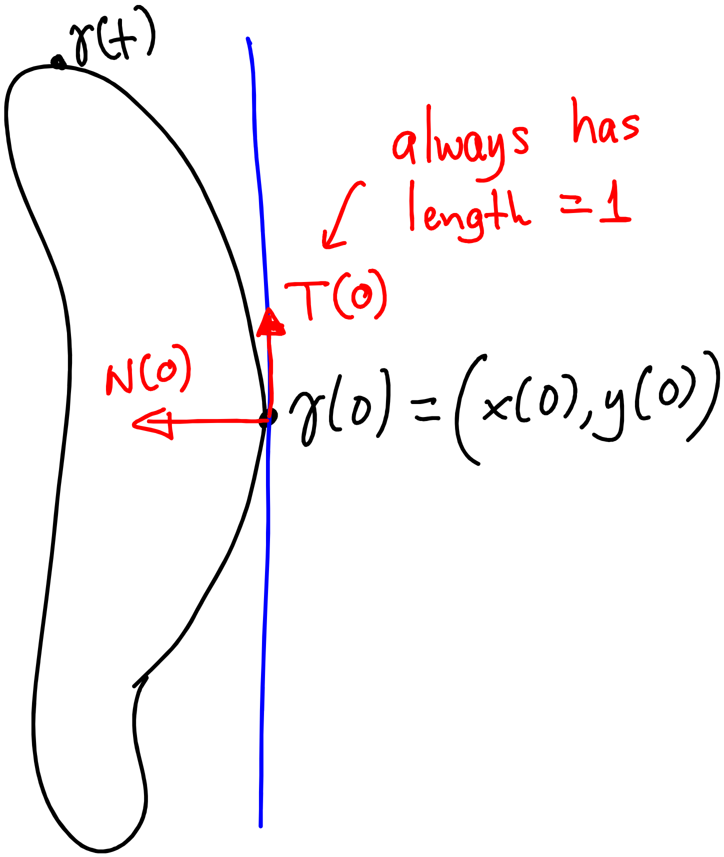
This is a very useful property of arc-length parameterized curves, because the tangent -estimated as the derivative of the curve- is always a unit-tangent!

The Unit Normal Vector

Let's look at the normal vector now



The Unit Normal Vector

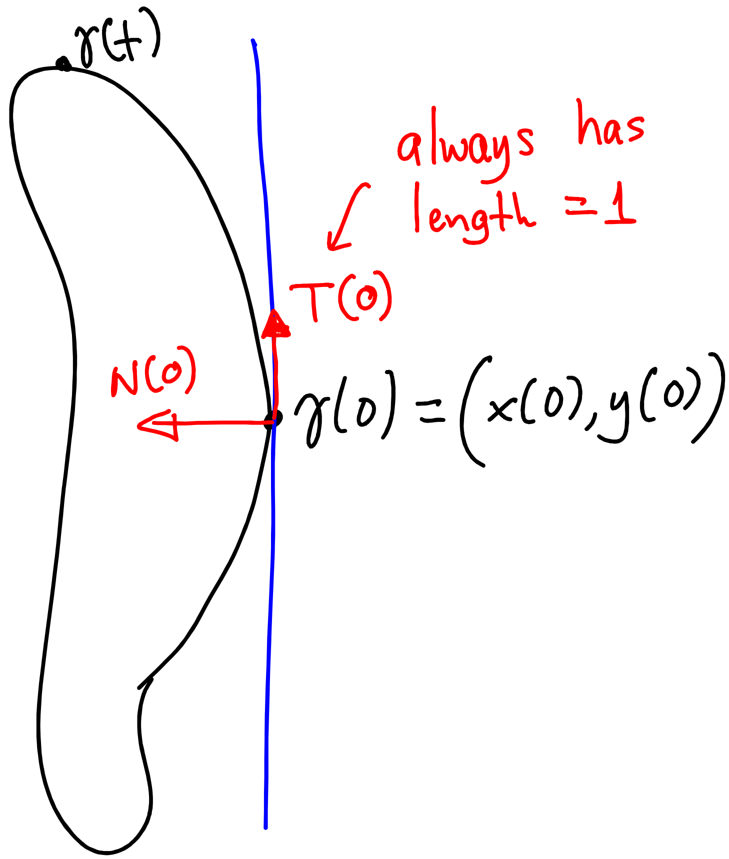


Today we learnt that the unit tangent is

$$T(t) = \frac{d\gamma}{dt}(t) \cdot \frac{1}{\left\| \frac{d\gamma}{dt}(t) \right\|_2}$$

How do we estimate the Unit Normal?

The Unit Normal Vector

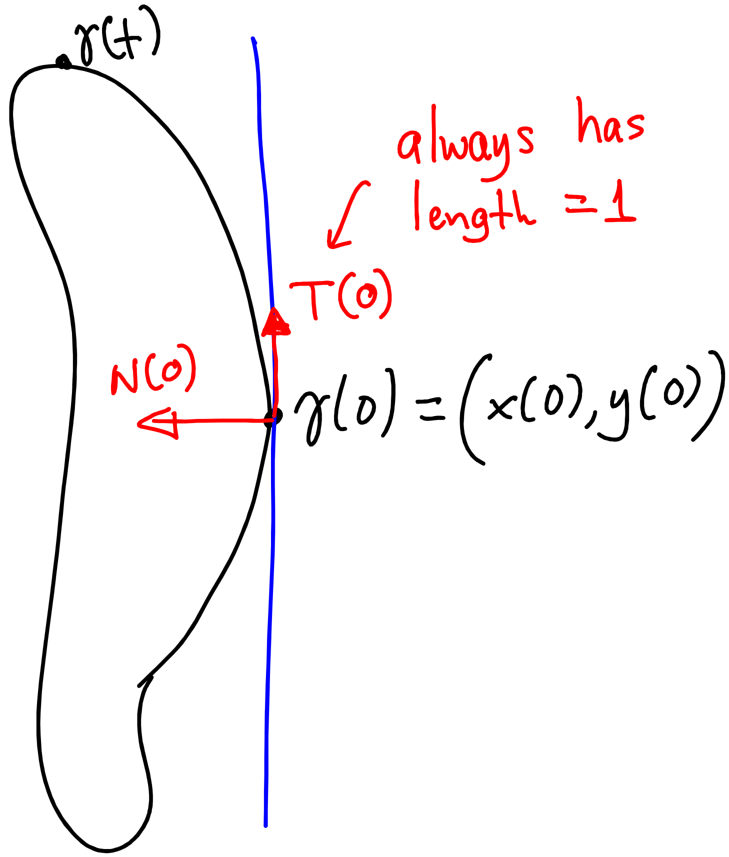


Today we learnt that the unit tangent is

$$T(t) = \frac{d\gamma}{dt}(t) \cdot \frac{1}{\left\| \frac{d\gamma}{dt}(t) \right\|_2}$$

As the orthogonal vector to T(t).
The (unit) normal vector N(t) is the counter-clockwise rotation of T(t) by 90 degrees.

The Unit Normal Vector



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As the orthogonal vector to T(t).
The (unit) normal vector N(t) is the counter-clockwise rotation of T(t) by 90 degrees.

$$N(t) = \frac{1}{\left\| \left(\frac{dy}{dt}, \frac{dx}{dt} \right) \right\|_2} \left(-\frac{dy}{dt}(t), \frac{dx}{dt}(t) \right)$$

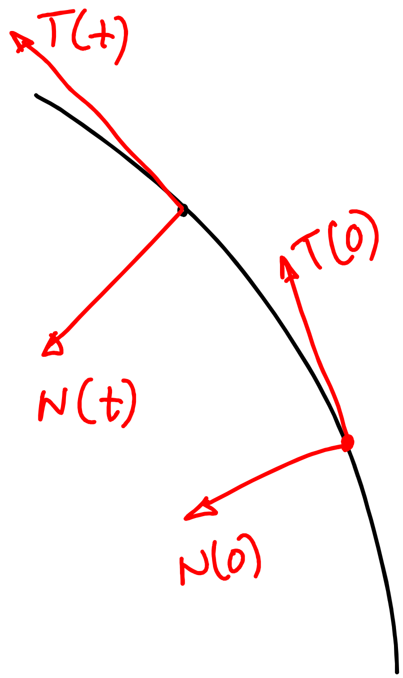
The Unit Normal Vector

Aside: what are orthogonal vectors?

Vectors **a** and **b** are orthogonal if and only if their dot product is zero. So if **a** = [a_x, a_y], and **b** = [b_x, b_y], then **a** and **b** are orthogonal if and only if:

$$\begin{pmatrix} a_x & a_y \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix} = a_x b_x + a_y b_y = 0$$

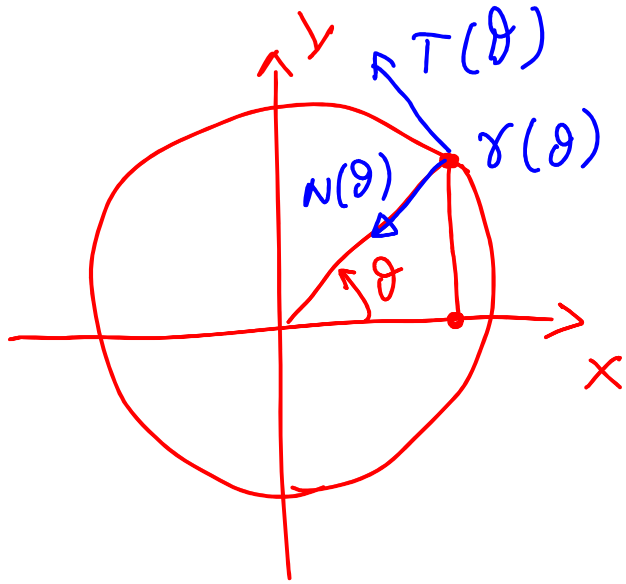
The Moving Frame



Putting the unit Tangent and the unit Normal together we get:
The Moving Frame, defined as the pair of orthogonal vectors $\{T(t), N(t)\}$

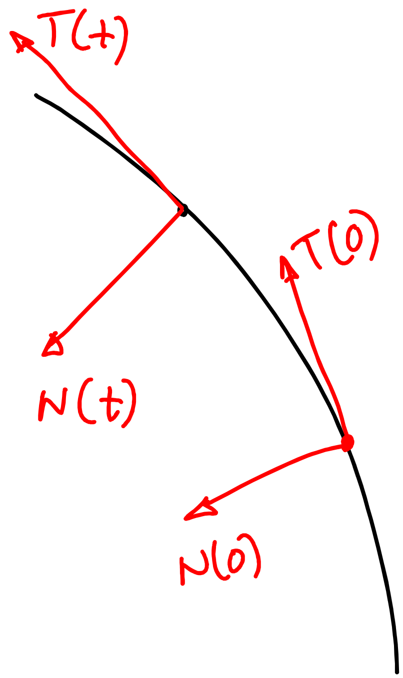
The Moving Frame

For example, the circle



$$\gamma(\vartheta) = \rho(\cos\vartheta, \sin\vartheta)$$
$$T(\vartheta) = (-\sin\vartheta, \cos\vartheta)$$
$$N(\vartheta) = (-\cos\vartheta, -\sin\vartheta)$$

The Moving Frame



Noteworthy:

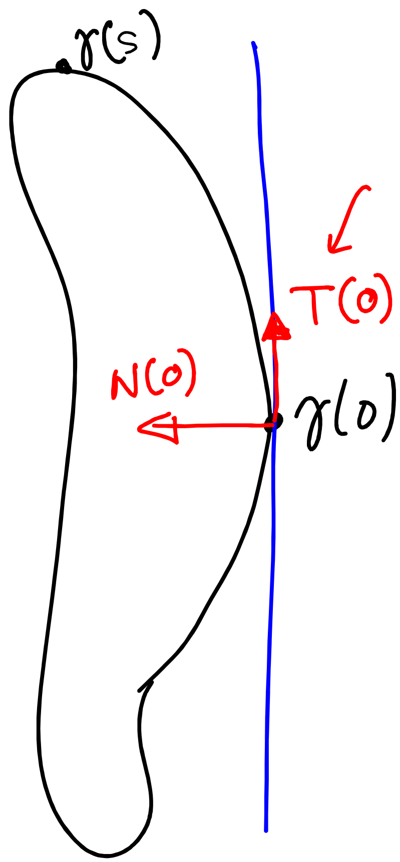
1. As we change the parameter t, the moving frame rotates
2. The faster the frame rotates, the more “curved” the curve is
3. The speed at which the moving frame is rotating can be estimated using a 1st order Taylor-series near t=0.

Topic 4.2:

Local analysis of 2D curve patches

- Representing 2D image curves
- Estimating differential properties of 2D curves
  - Tangent & normal vectors
  - The arc-length parameterization of a 2D curve
  - The curvature of a 2D curve

Arc-Length Parameterization: T(s) & N(s)



We know that:

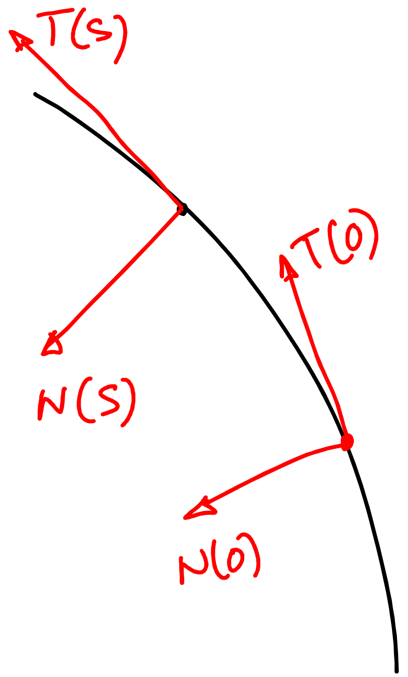
The unit tangent is: $T(s) = \frac{d\gamma}{ds}(s)$

And that the unit normal is the 90-deg counter-clockwise rotation:

$$N(s) = \left(-\frac{dy}{ds}(s), \frac{dx}{ds}(s)\right)$$

Note that we use "s" as the parameter to denote arc-length parameterizations. And we arc-length parameterizations because the expressions are simpler (see last slide of this lecture for comparison).

Defining the Curvature at a Point



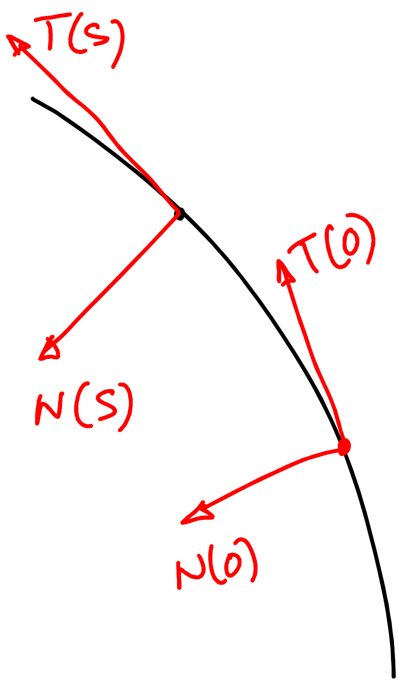
Theorem. Definition of curvature.

If s is the arc-length of a curve, then

$$\frac{dT}{ds}(s) = k(s) \cdot N(s)$$

$$\frac{dN}{ds}(s) = -\underbrace{k(s)}_{\substack{\text{curvature} \\ \text{(a scalar)}}} \cdot \underbrace{T(s)}_{\substack{\text{tangent} \\ \text{(a vector)}}}$$

Defining the Curvature at a Point



$$\frac{dT}{ds}(s) = k(s) \cdot N(s)$$

$$\frac{dN}{ds}(s) = -k(s) \cdot T(s)$$

The traditional way of writing the 1st order Taylor approximation of the moving frame is:

$$\{T(s), N(s)\} = \{T(0), N(0)\} + \left\{s \cdot \frac{dT}{ds}(0),\ s \frac{dN}{ds}(0)\right\}$$

But if we use the curvature k(s), it becomes

$$\{T(s), N(s)\} = \{T(0), N(0)\} + \{s \cdot k(0) \cdot N(0),\ - s \cdot k(0) \cdot T(0)\}$$

Defining the Curvature at a Point

The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

Proof of theorem

- Let $T(s) = (u(s), v(s))$ for some $u(\ ), v(\ )$

Defining the Curvature at a Point

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Proof of theorem

- Let $T(s) = (u(s), v(s))$ for some $u(\ ), v(\ )$
- Then $\frac{dT}{ds}(s) = \left(\frac{du}{ds}(s), \frac{dv}{ds}(s)\right)$

Defining the Curvature at a Point

The 1st order Taylor-series approximation becomes:

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Proof of theorem

- Let $T(s) = (u(s), v(s))$ for some $u(\ ), v(\ )$
- Then $\frac{dT}{ds}(s) = \left(\frac{du}{ds}(s), \frac{dv}{ds}(s)\right)$
- Length of $T(s) = 1 \ \forall s \Rightarrow$
  derivative of $(\text{length})^2 = 0 \Rightarrow$

Defining the Curvature at a Point

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- Let $T(s) = (u(s), v(s))$ for some $u(\ ), v(\ )$
- Then $\frac{dT}{ds}(s) = \left(\frac{du}{ds}(s), \frac{dv}{ds}(s)\right)$
- Length of $T(s) = 1 \ \forall s \Rightarrow$
  derivative of $(\text{length})^2 = 0 \Rightarrow$
- $\frac{d}{ds}\left(u^2(s) + v^2(s)\right) = 0 \Rightarrow$

Defining the Curvature at a Point

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- Then $\frac{dT}{ds}(s) = \left(\frac{du}{ds}(s), \frac{dv}{ds}(s)\right)$
- Length of $T(s) = 1 \ \forall s \Rightarrow$
  derivative of $(\text{length})^2 = 0 \Rightarrow$
- $\frac{d}{ds}\left(u^2(s) + v^2(s)\right) = 0 \Rightarrow$
- $2u(s) \cdot \frac{du}{ds}(s) + 2v(s) \cdot \frac{dv}{ds}(0) = 0 \Rightarrow$

Defining the Curvature at a Point

The 1st order Taylor-series approximation becomes:

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- Length of $T(s) = 1\ \forall s \Rightarrow$
  derivative of $(\text{length})^2 = 0 \Rightarrow$
- $\frac{d}{ds}\left(u^2(s) + v^2(s)\right) = 0 \Rightarrow$
- $2u(s) \cdot \frac{du}{ds}(s) + 2v(s) \cdot \frac{dv}{ds}(0) = 0 \Rightarrow$

$$\begin{bmatrix} \frac{du}{ds}(s) & \frac{dv}{ds}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = 0$$

Defining the Curvature at a Point

The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

Proof of theorem

- Let $T(s) = (u(s), v(s))$ for some $u(\ ), v(\ )$
- Then $\frac{dT}{ds}(s) = \left(\frac{du}{ds}(s), \frac{dv}{ds}(s)\right)$
- Length of $T(s) = 1 \ \forall s \Rightarrow$ derivative of $(\text{length})^2 = 0 \Rightarrow$
- $\frac{d}{ds}\left(u^2(s) + v^2(s)\right) = 0 \Rightarrow$
- $2u(s) \cdot \frac{du}{ds}(s) + 2v(s) \cdot \frac{dv}{ds}(0) = 0 \Rightarrow$

$$\frac{dT}{ds}(s) \rightsquigarrow \begin{bmatrix} \frac{du}{ds}(s) & \frac{dv}{ds}(s) \end{bmatrix} \underbrace{\begin{bmatrix} u(s) \\ v(s) \end{bmatrix}}_{\leftarrow T(\ )} = 0$$

Defining the Curvature at a Point

The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

$$\frac{dT}{ds}(s) \sim \begin{bmatrix} \frac{du}{ds}(s) & \frac{dv}{ds}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ v(s) \end{bmatrix} = 0 \leftarrow T(\ )$$

Defining the Curvature at a Point

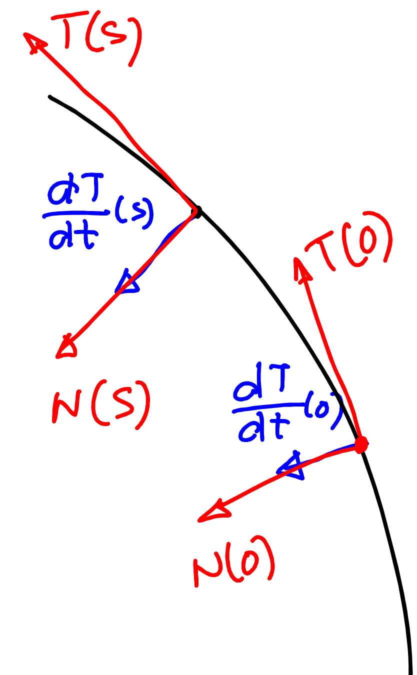
The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t\cdot k(0)\cdot N(0), -t\cdot k(0)\cdot T(0)\}$$

$$\frac{dT}{ds}(s) \rightsquigarrow \begin{bmatrix}\frac{du}{ds}(s) & \frac{dv}{ds}(s)\end{bmatrix}\begin{bmatrix}u(s)\\ v(s)\end{bmatrix} = 0 \Rightarrow \quad \leftarrow T(\ )$$

$\frac{dT}{ds}(s)$ is orthogonal to $T(s)$ $\Rightarrow$

$\frac{dT}{ds}(s)$ is collinear with $N(s)$



And the scaling constant is k(s)

$$\Downarrow$$

$$\frac{dT}{ds}(s) = k(s)\cdot N(s)$$

Defining the Curvature at a Point

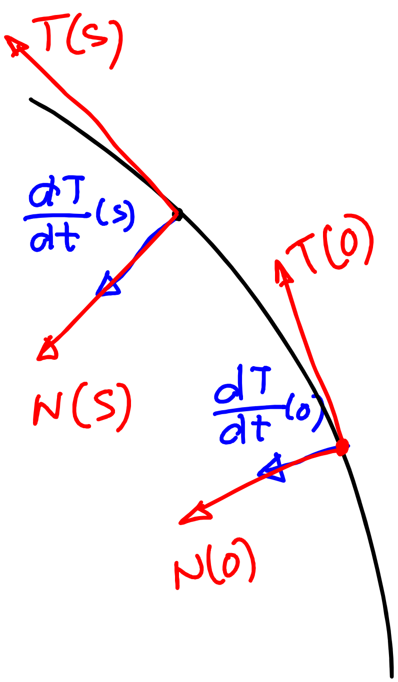
The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

And the scaling constant is k(s)

$$\frac{dT}{ds}(s) = k(s) \cdot N(s)$$

What is this constant saying?



Defining the Curvature at a Point

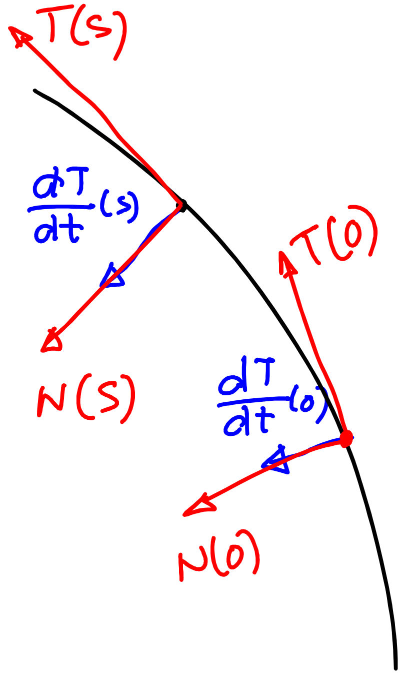
The 1st order Taylor-series approximation becomes:

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

And the scaling constant is k(s)

$$\frac{dT}{ds}(s) = k(s) \cdot N(s)$$

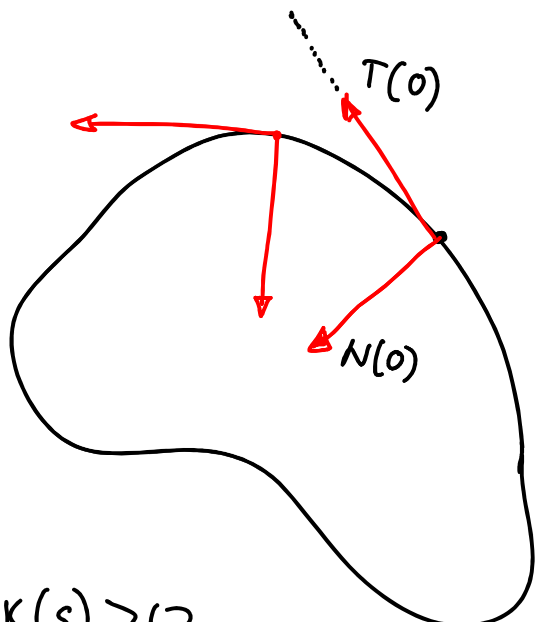
What is this constant saying?



How much of the Normal do we need to add to the Tangent T(0) to approximate the tangent at T(t).

Interpreting the Sign of the Curvature k(s)

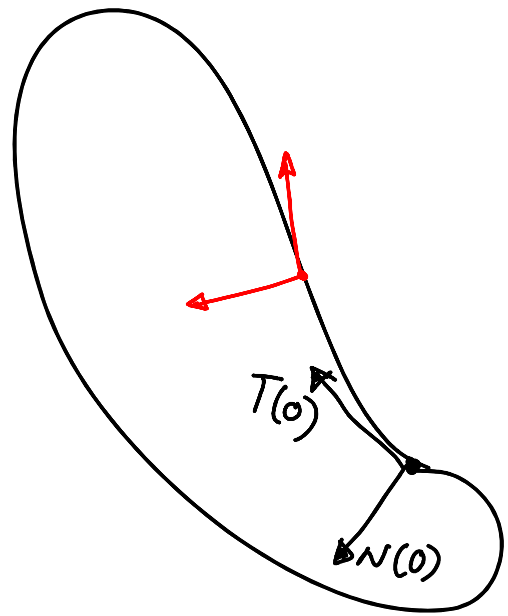
$$\frac{dT}{ds}(s) = k(s) \cdot N(s)$$



$K(s) > 0$

curve bends
in the direction
of the normal

$$\frac{dN}{ds}(s) = -k(s) \cdot T(s)$$



$K(s) < 0$

curve bends
in the opposite
direction from
the normal

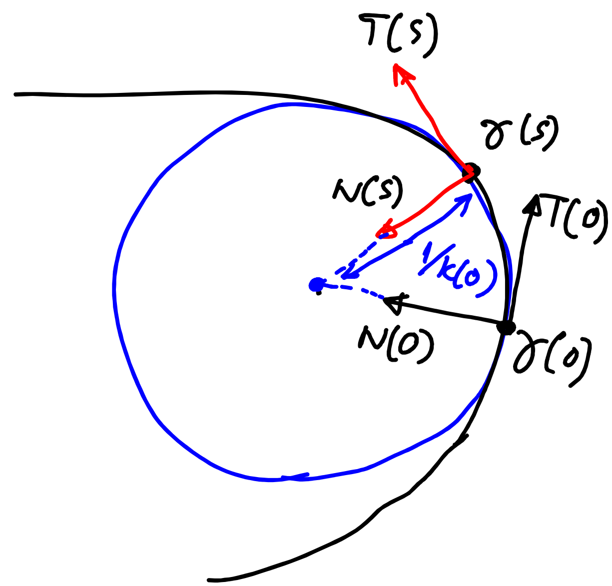
Interpreting the Absolute Value of k(s)

$$\{T(t), N(t)\} = \{T(0), N(0)\} + \{t \cdot k(0) \cdot N(0), -t \cdot k(0) \cdot T(0)\}$$

What is the intuition of the above equation then?

The equation is saying, look we can approximate γ(s) (by approximating the Tangent and the Normal) using a circle that:

- passes through γ(0),
- is tangent to T(0), and
- passes through γ(s)



- The radius of the circle is $1/k(s)$

The Arc-Length Parameterization & k(s)

Example: the curvature of a circle of radius r.

Parametric equation: g(t) = r (cos t, sin t).

Arc-length parameterization $\gamma(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$

First derivative $\frac{d\gamma}{ds}(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right) = T(s)$

Second derivative $\frac{dT}{ds}(s) = \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right) = \frac{1}{r} N(s)$

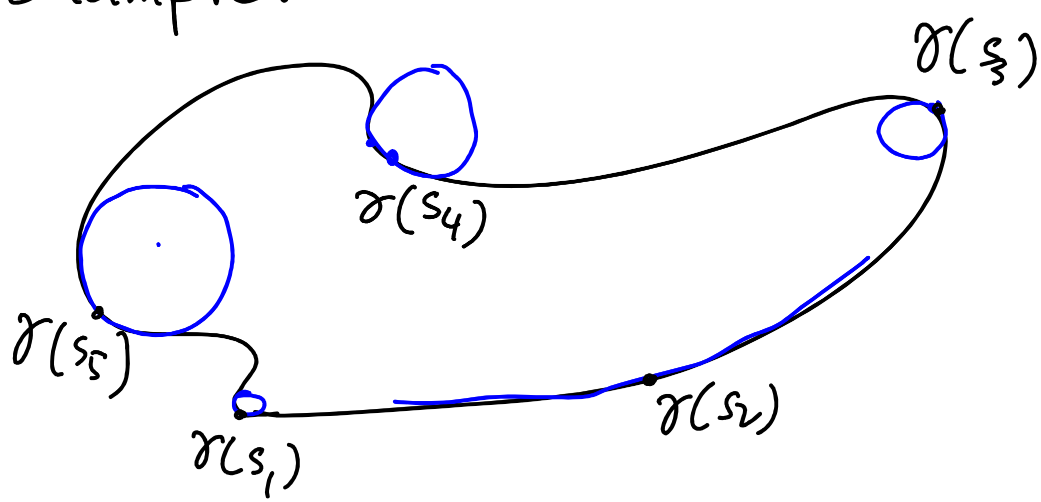
curvature of the circle

The Circle of Curvature k(s)

Definition (Circle of curvature at $\gamma(s)$)

The circle of radius $1/k(s)$ that passes through $\gamma(s)$ and is tangent to $T(s)$

Example:



k(t) for Non-Arc-Length Parameterizations

Prove that if $\gamma(t) = (x(t), y(t))$ the curvature at $\gamma(t)$ is given by the formula

$$\boxed{k(t) = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}}$$

The 1st and 2nd derivatives of $x(t), y(t)$ are extremely informative about the curve's shape!!