Input Convex Gradient Networks

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Motivation

- From Brenier's theorem [2], we know that for two measures $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$, such that μ does not give mass to small sets, there exists a convex potential φ such that

$$\nabla \varphi = \underset{T : T_{\#}\mu = \nu}{\operatorname{argmin}} \int_{\mathbb{R}^n} |x - T(x)|^2 d\mu$$

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- This result motivates use of convex gradients for estimating OT maps [5], more recently for applications such as Wasserstein gradient flows,[4] density estimation, generative modelling [3]...

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- To constraint N_θ to parameterize a convex gradient, apply the following theorem

Theorem (3 in paper)

For any smooth $G : \mathbb{R}^n \to \mathbb{R}^n$ such that DG_x is symmetric PSD for all x, there exists a convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $G = \nabla g$. i.e G is the gradient of convex function.

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- Since the condition we want to enforce is on the Jacobian, we work with the Jacobian (or Hessian of the potential) then integrate.

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- So given a suitable network $M_\theta : \mathbb{R}^n \to \mathbb{R}^m$, we compute

$$N_\theta(x) = \int_0^1 [D(M_\theta)_{sx}]^T D(M_\theta)_{sx} x ds$$

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  - **Current limitation**: can only use one layer hidden networks.

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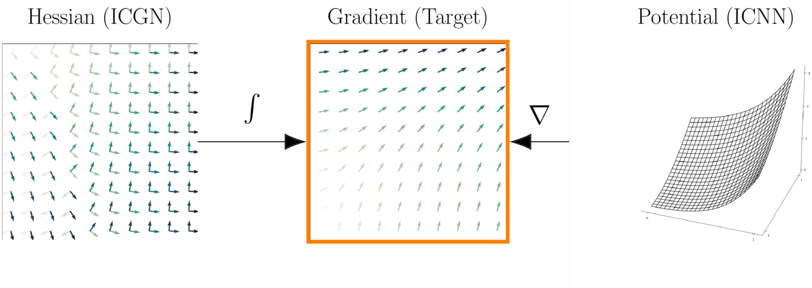
Takeaway: ICGN vs ICNN for modelling gradients

As a comparison from our approach to the ICNN gradient:



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