We generalize Ridge Rider to games with the following:

- The Hessian’s generalization for games – i.e., the Game Hessian – may have complex EVals from a lack of symmetry. This gives non-conservative dynamics.
- In many games – ex., the Iterated Prisoner’s Dilemma (IPD) – we are no longer be in a conservative gradient field, allowing more solution and bifurcation types.

**Generalizing the Hessian for Games**

\[
\text{Game Hessian} \quad \hat{H} = \begin{bmatrix}
\text{Player A Hessian} \nabla_\theta^2 \mathcal{L}_A \\
\nabla_\theta \nabla_\theta^\top \mathcal{L}_B \\
\end{bmatrix}
\]

- **New Toy Problems**
  - Matching Pennies is a 2 param. game with imaginary EVals, but only 1 solution.
  - Small IPD is a 2 param. IPD with TT and DD solutions, but only real EVals.
  - Mixing these gives a 2 param. problem like the full IPD with multiple solutions, complex EVals, and a Hopf bifurcation.

**Applying our Method on Toy Problems**

- For both the small IPD (left) and mixed objective (right) our method – Game Ridge Rider (GRR) – finds all solutions.

**Finding Diverse Solutions in the Iterated Prisoners Dilemma (IPD)**

- Randomly initializing then applying a training method only finds 1 solution mode.
- Our method finds both solution modes.
- If we don’t start at a saddle, then branching doesn’t affect the solution.

**Takeaways**

- Differentiable games generalize minimization, but with non-conservative dynamics from complex EVals.
- We can view methods for diverse solutions in minimization – i.e., Ridge Rider (RR) – as finding bifurcations in conservative systems and branching.
- The viewpoint allows usage of tools from dynamical systems to generalize RR to non-conservative setups.
- Our method generalizes RR by branching at Hopf bifurcations and applying arbitrary optimizers after branching.

**Motivation**

- Finding different solution types has been useful in minimization - ex., shape vs. texture for CNNs.
- Differentiable games generalize minimization.
  \[
  \begin{align*}
  \theta^*_A &\in \arg\min_{\theta_A} \mathcal{L}_A(\theta_A, \theta_B), \\
  \theta^*_B &\in \arg\min_{\theta_B} \mathcal{L}_B(\theta_A, \theta_B)
  \end{align*}
  \]
- Games are increasingly important in ML – ex., GANs, hyperparameter optimization, self-play, meta-learning, adversarial examples, many others.
  - **Goal:** Find diverse solutions in differentiable games – ex., where players work together or battle each-other.

**Background**

- Ridge rider (RR) [2] finds diverse solution for a single objective by following negative EVals of the Hessian at saddles.
- The Hessian is symmetric with real EVals, so we have conservative dynamics.
- Bifurcations are where small changes cause solution differences.
- Saddle points are a key bifurcation in conservative systems.

**Our Method – Game Ridge Rider (GRR)**

We generalize Ridge Rider to games with the following:

- Complex EVals may have EVecs with complex entries. We use an EVec selection for conjugate pairs that has all real entries, so we can follow it.
- We detect and allow for branching at new types of bifurcations – ex., Hopf where the negative real part of an EVal crosses the imaginary axis.
- We apply an arbitrary optimization algorithm after branching.

**References**
