Epipolar Geometry & Stereo II

Essential & Fundamental Matrices, 8-point algorithm, active stereo

CSC420
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TCIG
Toronto Computational Imaging Group

Announcements

- HW 3 is out (due Nov 13)
- midterm on Oct 24 (Friday)!

Midterm Information (Oct 24)

Updated 27 seconds ago by David Lindell

Hi all,

The midterm for the course will be held on October 24th in lieu of the tutorial hour.

- Section LEC0101: SF 3202 1-2pm
- Section LEC0201: SF 3202 3-4pm

Please arrive promptly at the hour so that we can distribute the exam booklets and begin and end on time.

The exam will be a combination of multiple choice and long answer. You will write directly on the exam booklet, so bring a pencil and eraser. The exam is closed book and no notes are allowed.

Also, you must abide by the honor code — do not discuss the content of the midterm exam with any other students until after the grades have been released (obviously we have students taking the exam at different times, so please abide by this out of fairness to everyone).

Please comment on this post with any questions.

David

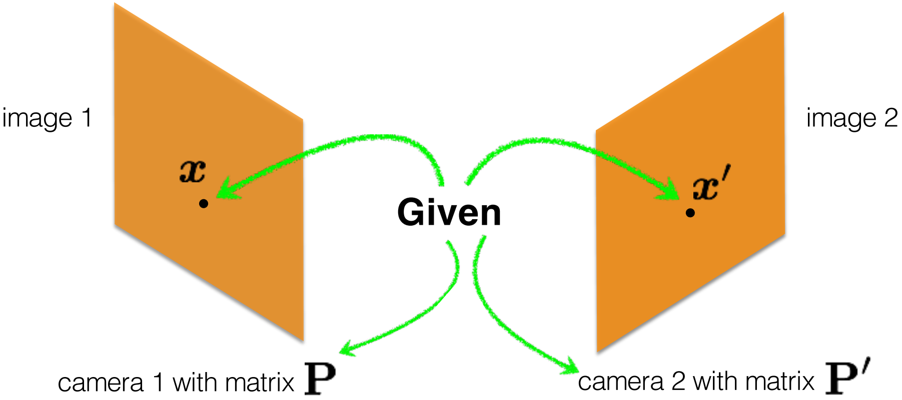
Course Overview

Overview

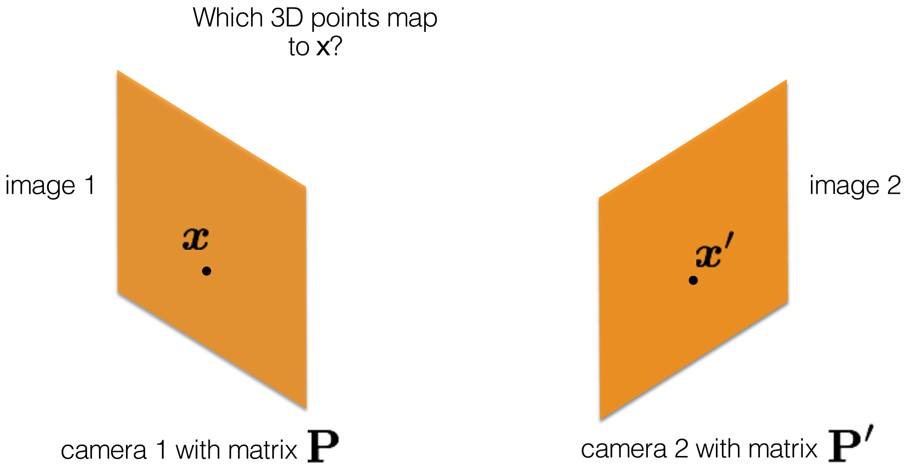
- Triangulation
- Epipolar geometry
- Essential matrix & Fundamental matrix
- 8-point algorithm
- Stereo rectification
- Stereo matching & structured light

Triangulation

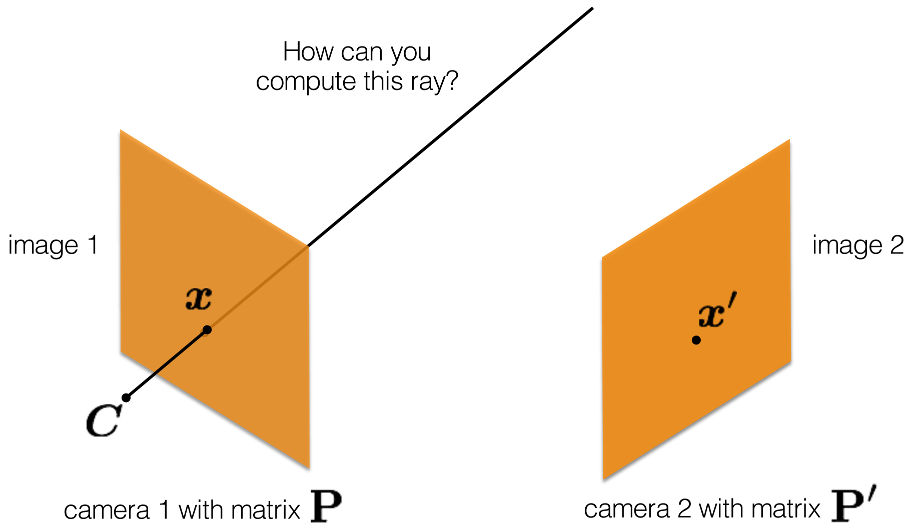
Triangulation



Triangulation



Triangulation

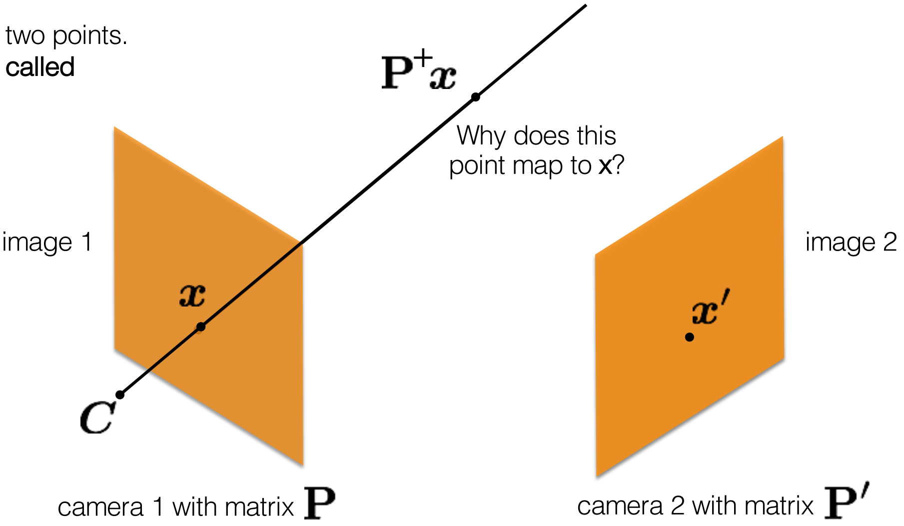


Triangulation

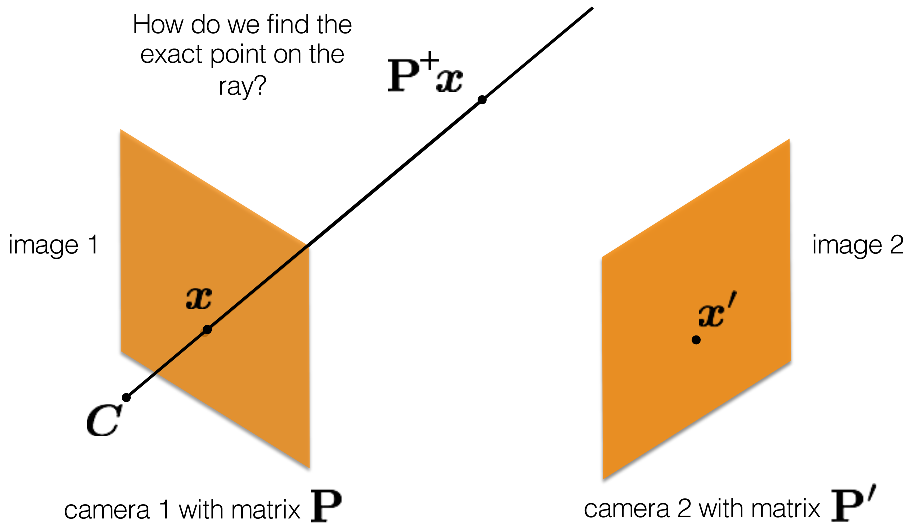
Create two points on the ray:
1) find the camera center; and
2) apply the pseudo-inverse of **P** on **x**.

Then connect the two points.

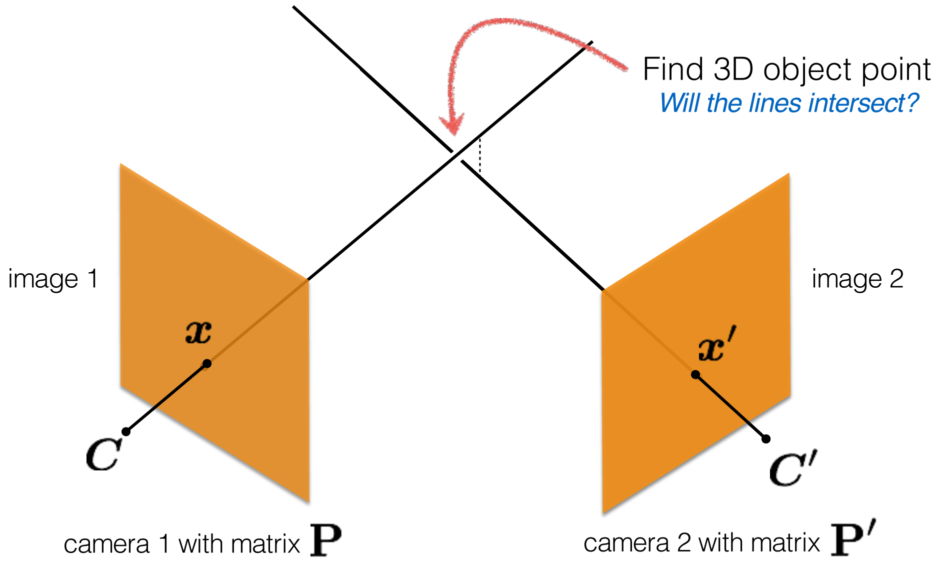
This procedure is called <u>**backprojection**</u>



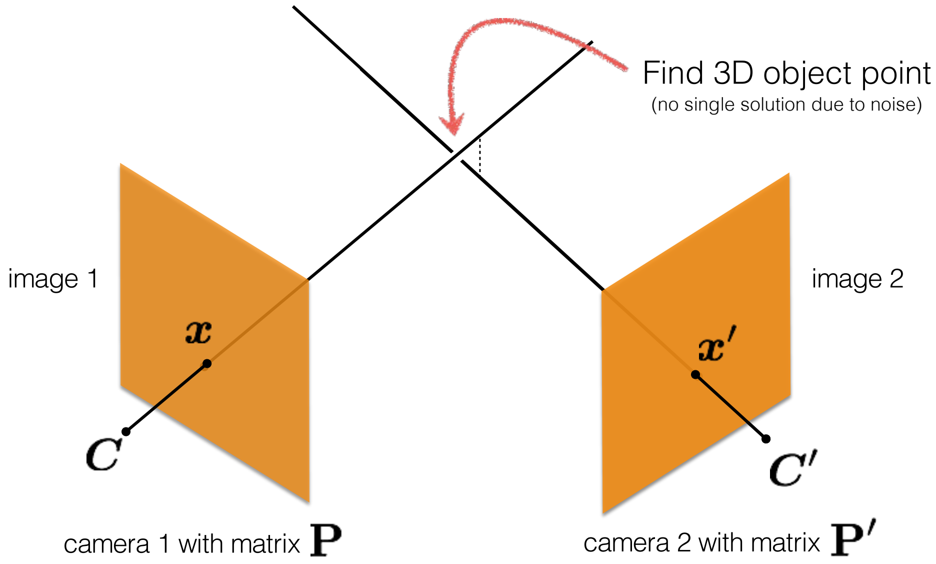
Triangulation



Triangulation



Triangulation



Triangulation

Given a set of (noisy) matched points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

and camera matrices

$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{X}$$

$$\underbrace{\mathbf{x}}_{\text{known}} = \underbrace{\mathbf{P}}_{\text{known}}\boldsymbol{X}$$

*Can we compute **X** from a single correspondence **x**?*

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

This is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homorogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(inhomogeneous coordinate)

Same ray direction but differs by a scale factor

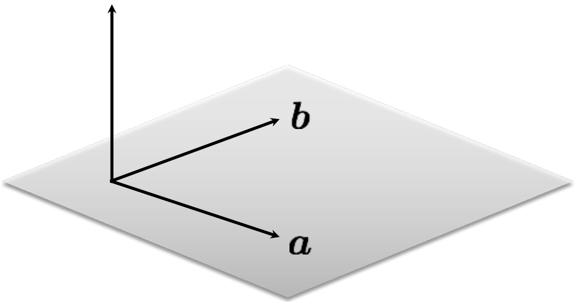
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

Linear algebra reminder: cross product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero vector

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

Linear algebra reminder: cross product

Cross product

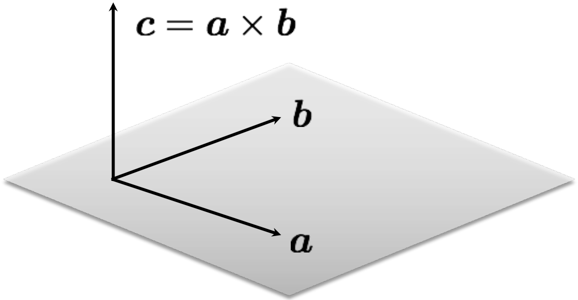
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Can also be written as a matrix multiplication

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Skew symmetric

Compare with: dot product



$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

dot product of two orthogonal vectors is (scalar) zero

Back to triangulation

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

How can we rewrite this using vector products?

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \boldsymbol{X} = \mathbf{0}$$

Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Do the same after first expanding out the camera matrix and points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{---} & \boldsymbol{p}_1^\top & \text{---} \\ \text{---} & \boldsymbol{p}_2^\top & \text{---} \\ \text{---} & \boldsymbol{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Do the same after first expanding out the camera matrix and points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} — & \boldsymbol{p}_1^\top & — \\ — & \boldsymbol{p}_2^\top & — \\ — & \boldsymbol{p}_3^\top & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you [redacted] equations

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Remove third row, and rearrange as system on unknowns

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

Now we can make a system of linear equations
(two lines for each 2D point correspondence)

Concatenate the 2D points from both images

Two rows from camera one

Two rows from camera two

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

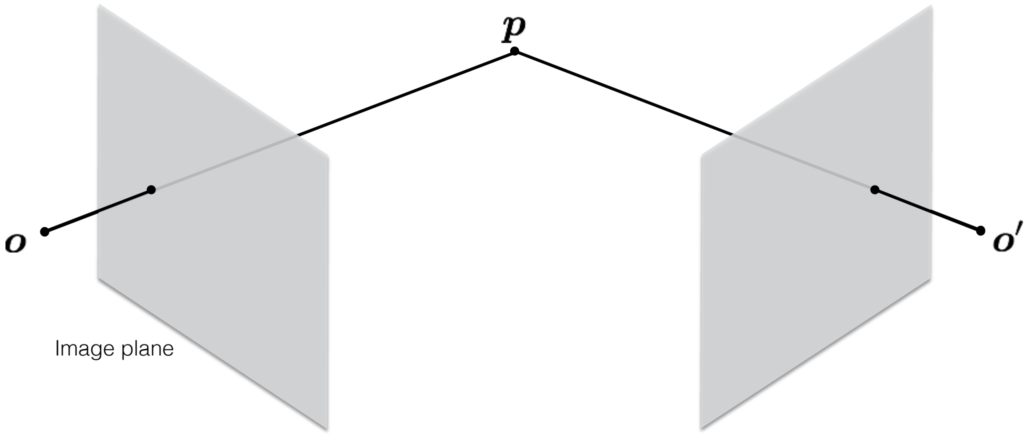
$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

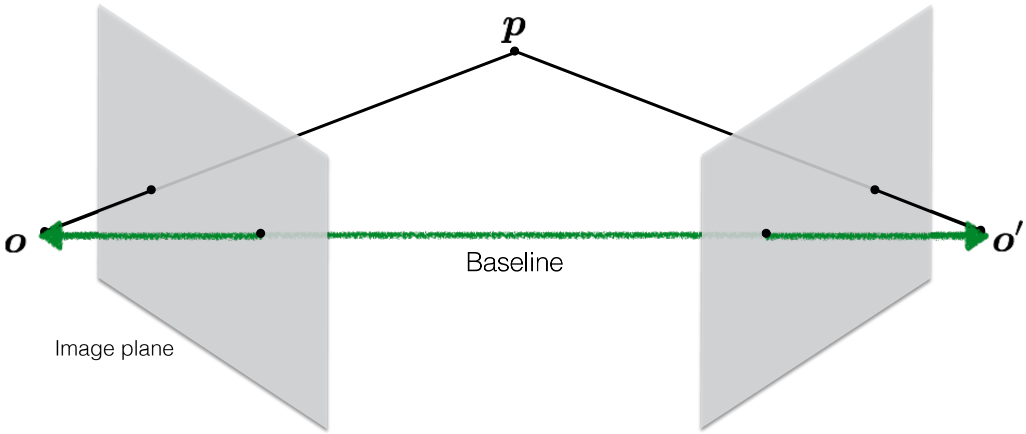
S V D !

Epipolar geometry

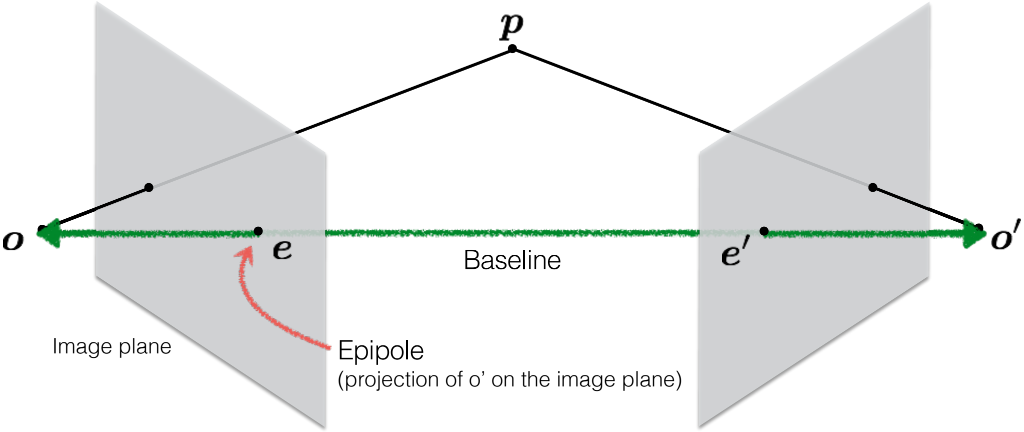
Epipolar geometry



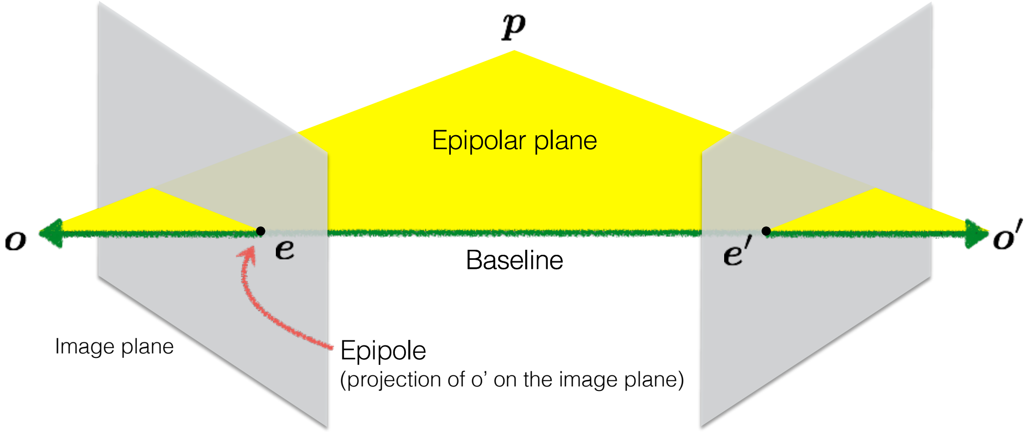
Epipolar geometry



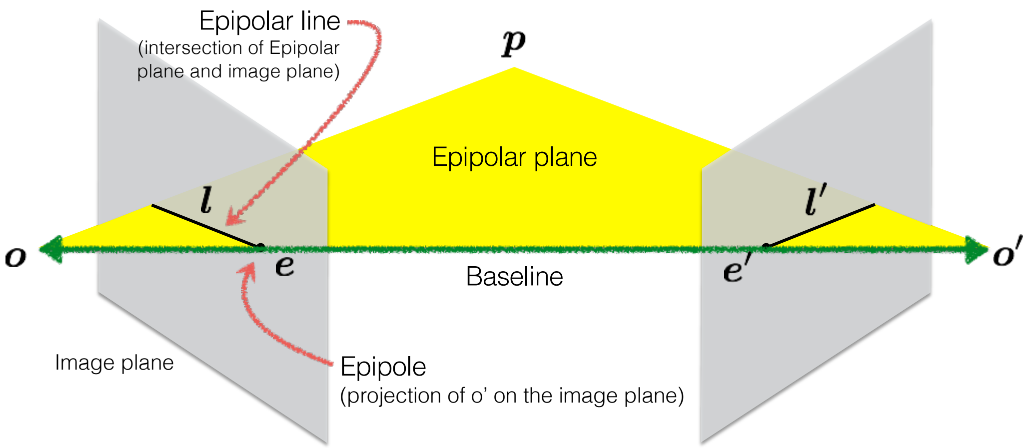
Epipolar geometry



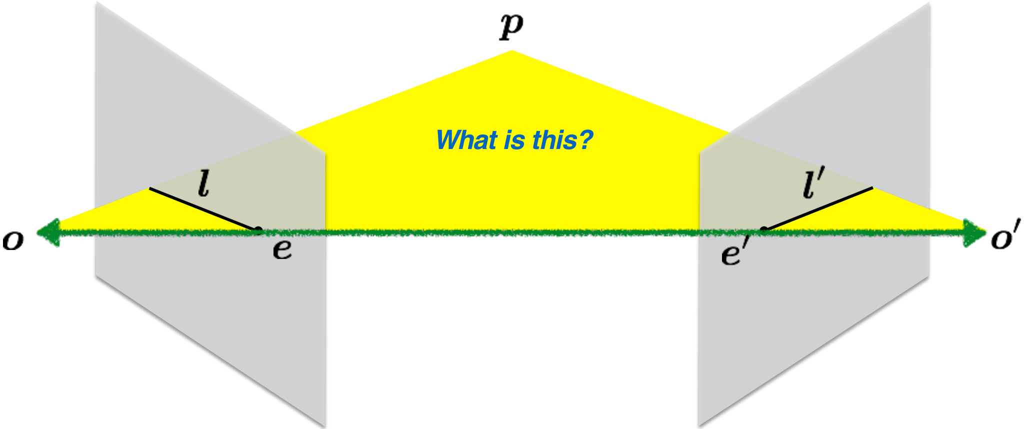
Epipolar geometry



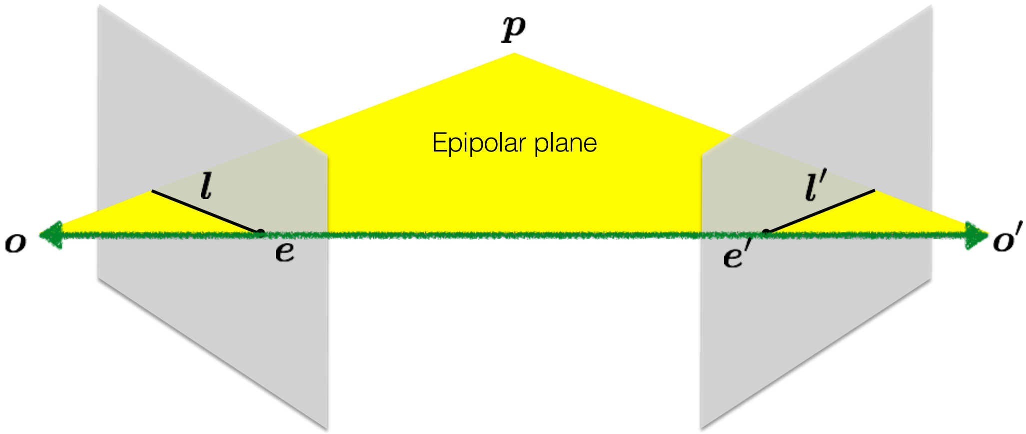
Epipolar geometry



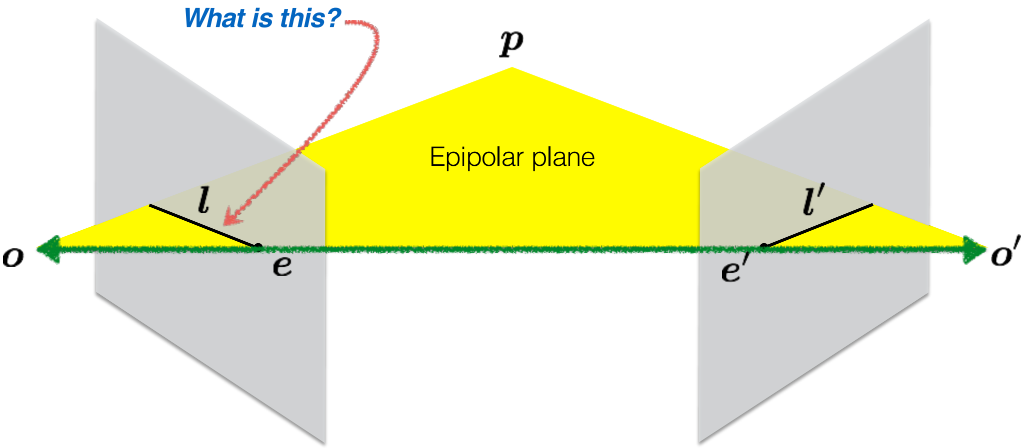
Quiz



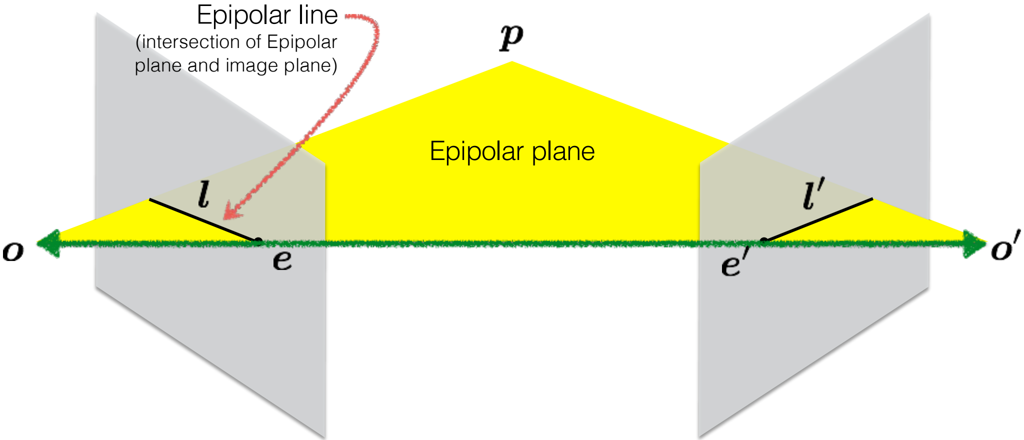
Quiz



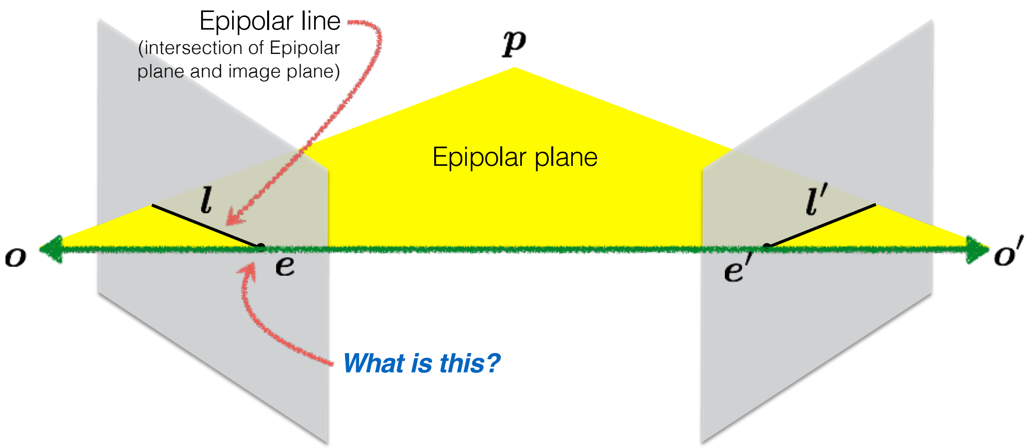
Quiz



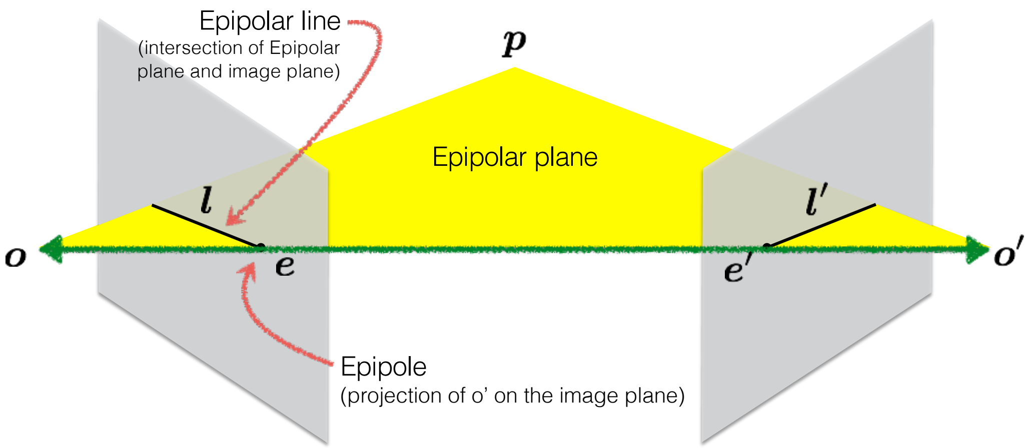
Quiz


Epipolar line
(intersection of Epipolar
plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'

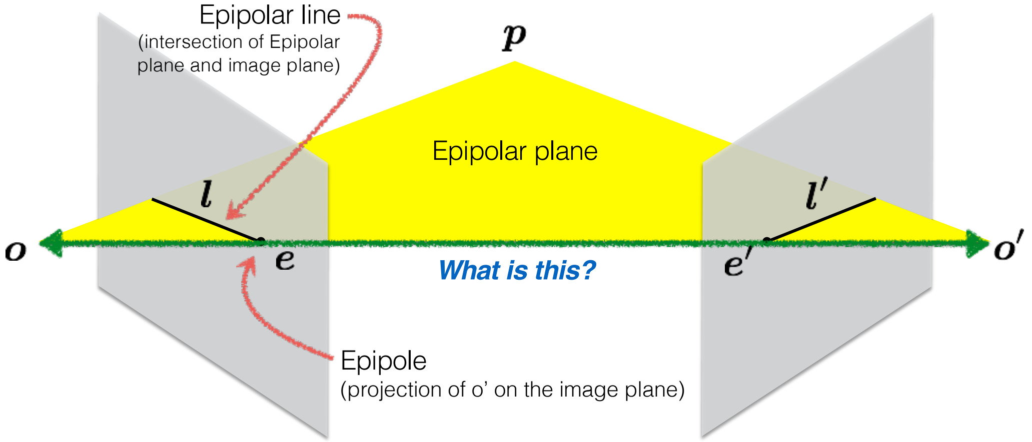
Quiz


Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'
What is this?

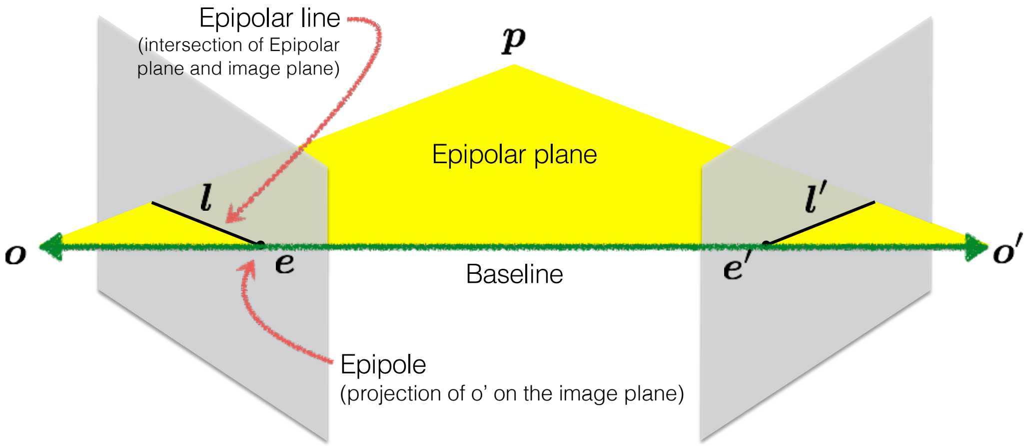
Quiz


Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'
Epipole
(projection of o' on the image plane)

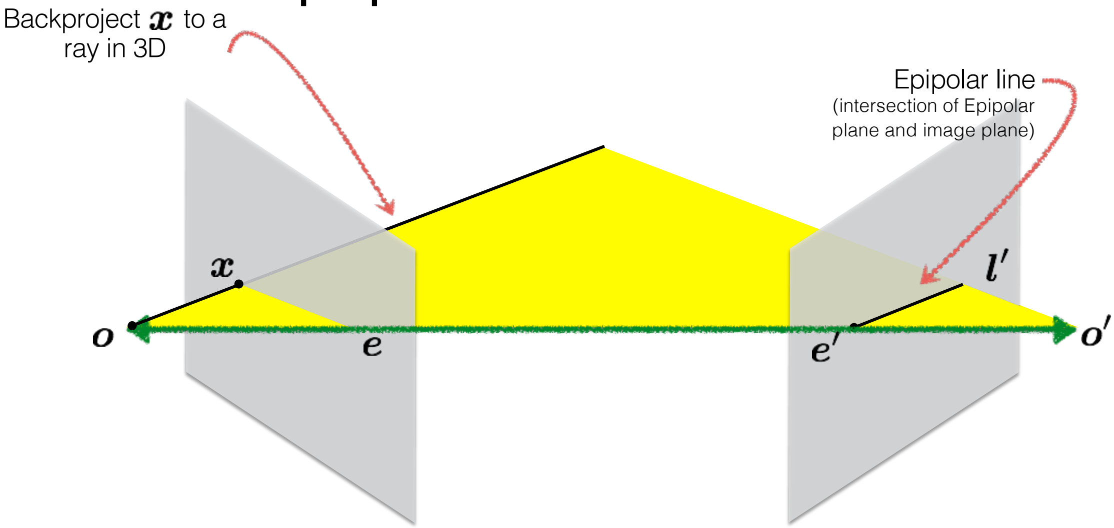
Quiz


Epipolar line
(intersection of Epipolar
plane and image plane)
p
Epipolar plane
l
l′
o
e
What is this?
e′
o′
Epipole
(projection of o’ on the image plane)

Quiz

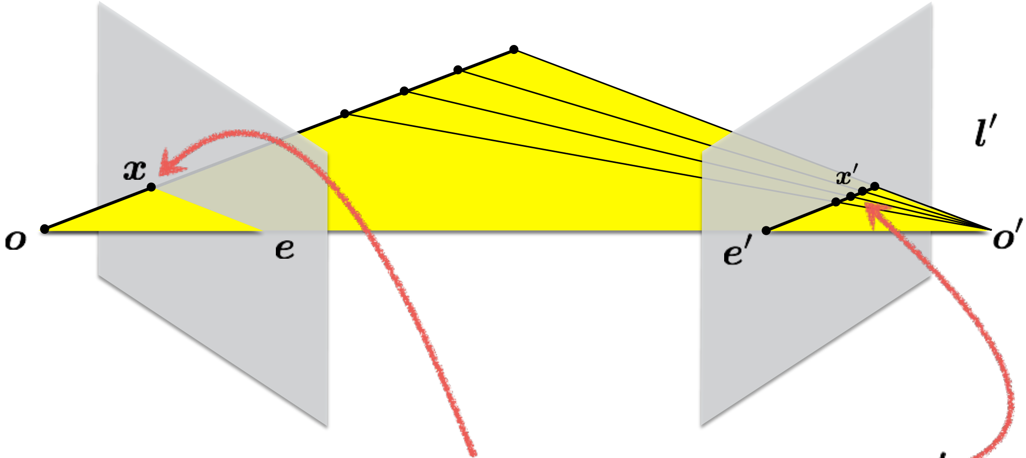

Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
Baseline
e'
o'
Epipole
(projection of o' on the image plane)

Epipolar constraint

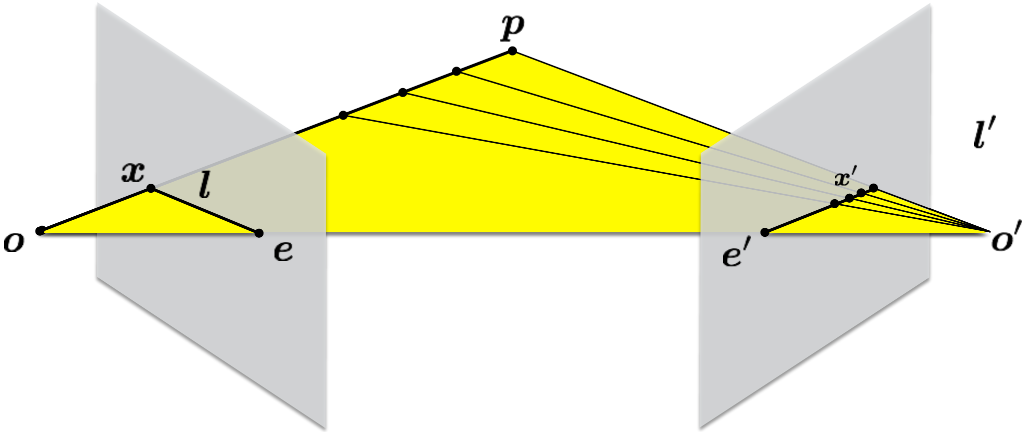


Another way to construct the epipolar plane, this time given $\boldsymbol{x}$

Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$



The point **x** (left image) maps to a ____________ in the right image

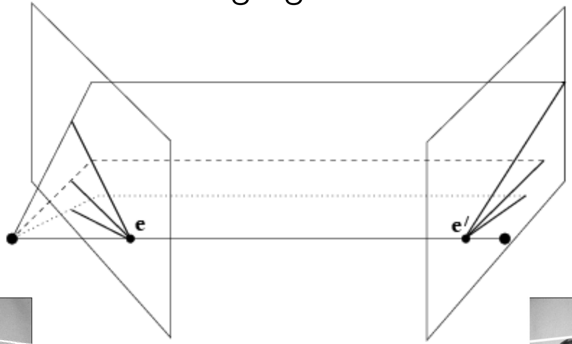
The baseline connects the ____________ and ____________

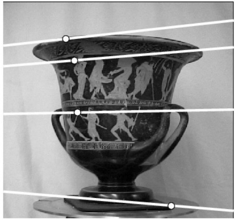
An epipolar line (left image) maps to a ___________ in the right image

An epipole **e** is a projection of the _______________ on the image plane

All epipolar lines in an image intersect at the _______________

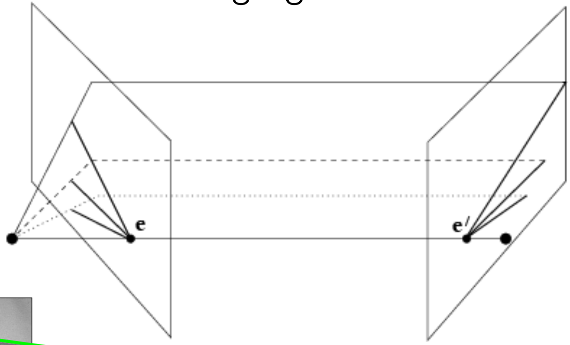
Converging cameras

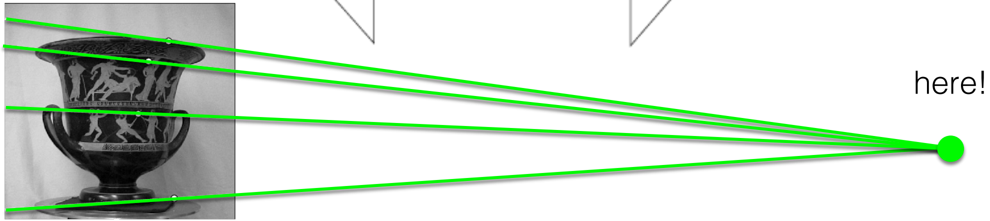




Where is the epipole in this image?

Converging cameras

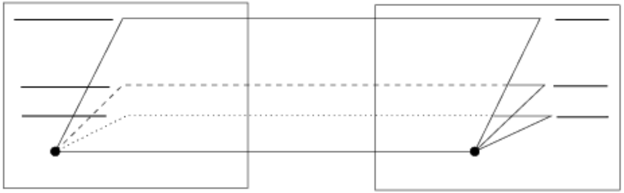


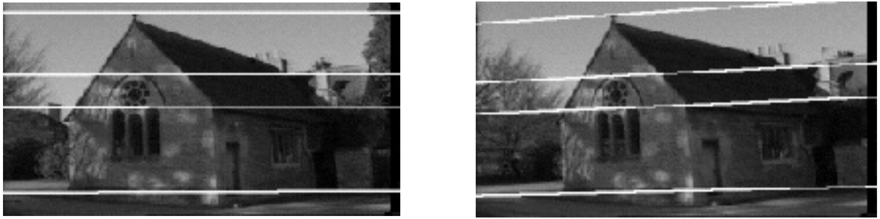


Where is the epipole in this image?

It’s not always in the image

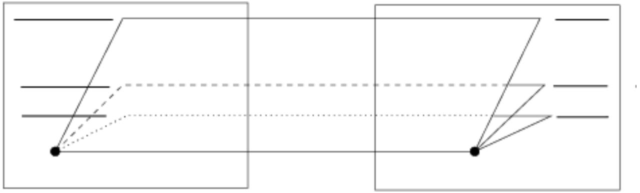
Parallel cameras

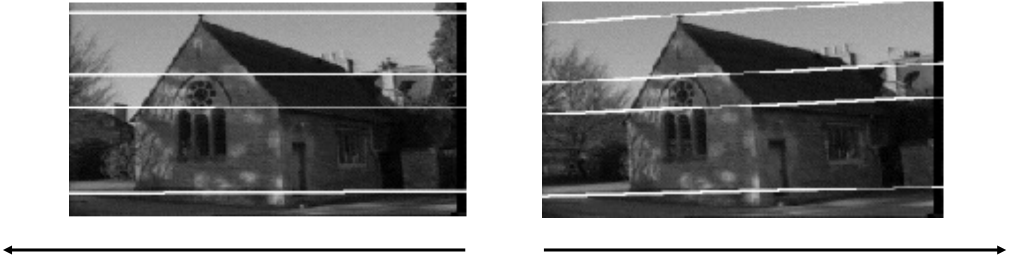




Where is the epipole?

Parallel cameras

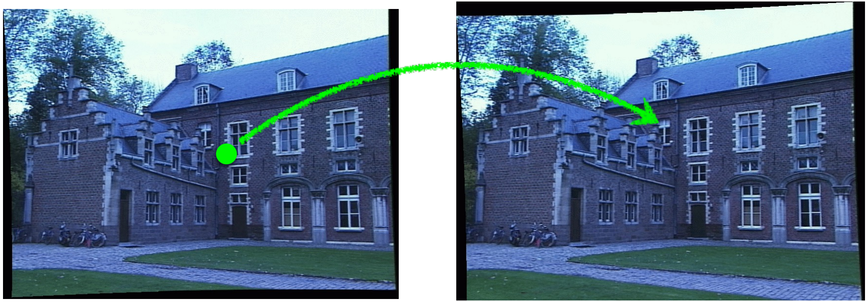




epipole at infinity

The epipolar constraint is an important concept for stereo vision

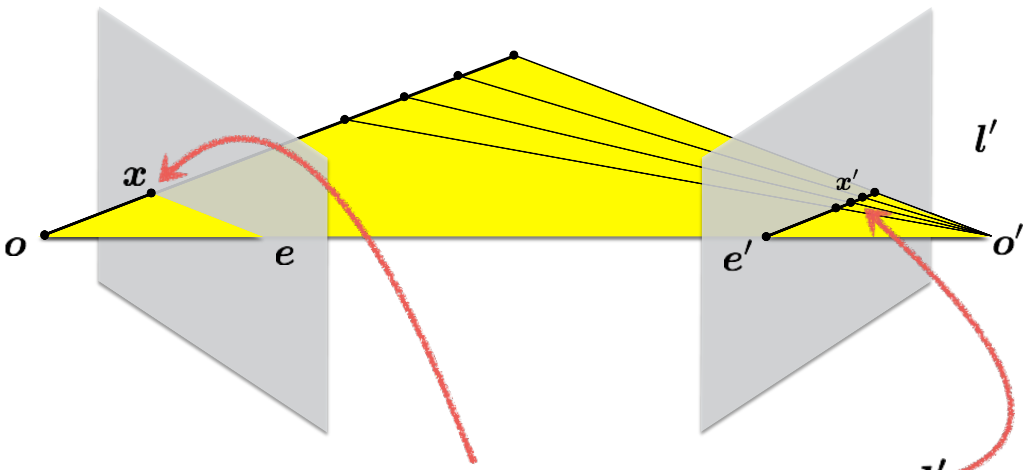
Task: Match point in left image to point in right image



Left image　　　　Right image

How would you do it?

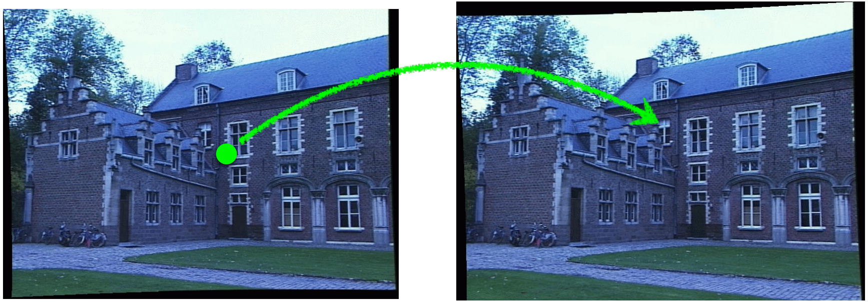
Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

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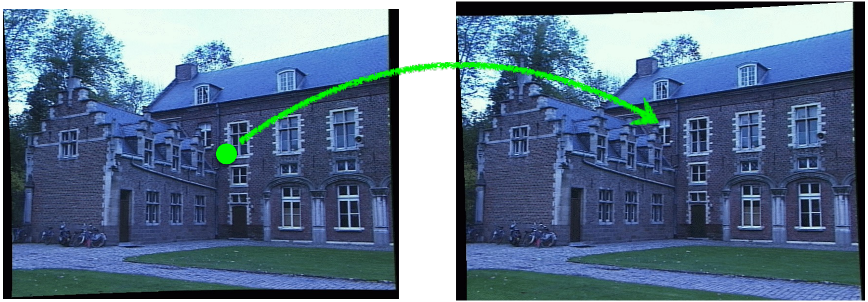


Left image

Right image

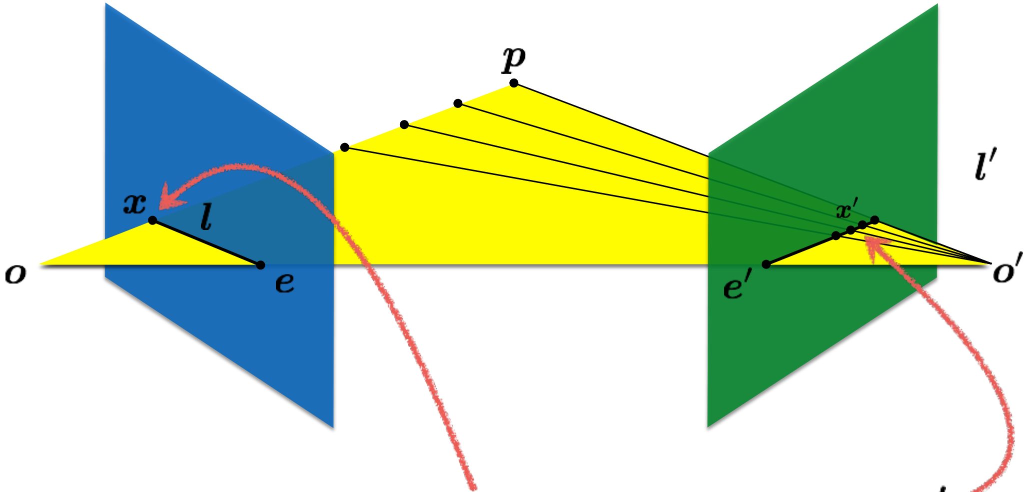
Want to avoid search over entire image

Epipolar constraint reduces search to a single line

How do you compute the epipolar line?

The essential matrix

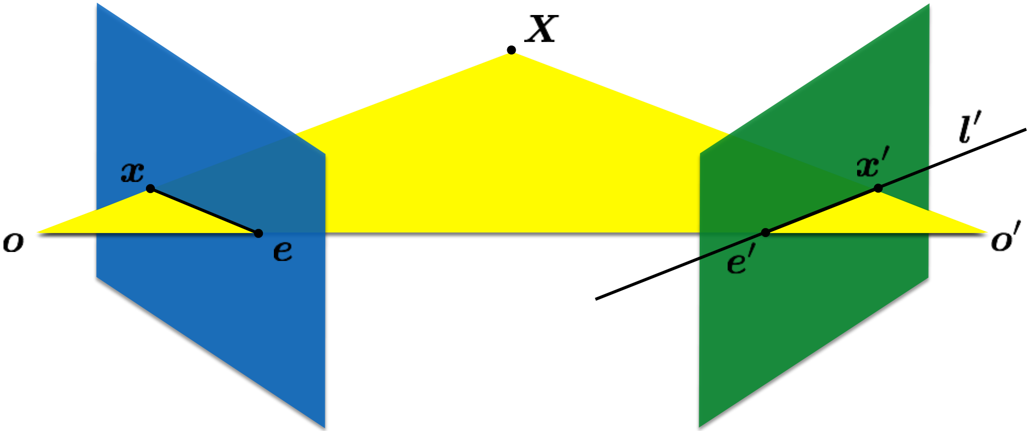
Recall:Epipolar constraint



Potential matches for x lie on the epipolar line l'

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$



Motivation

The Essential Matrix is a 3 x 3 matrix that
encodes **epipolar geometry**

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second image.

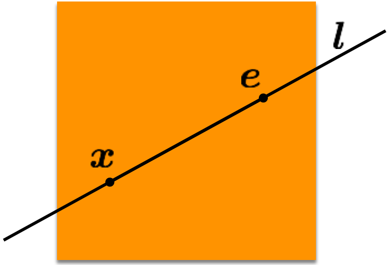
Representing the …

Epipolar Line

$$ax + by + c = 0$$

in vector form

$$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

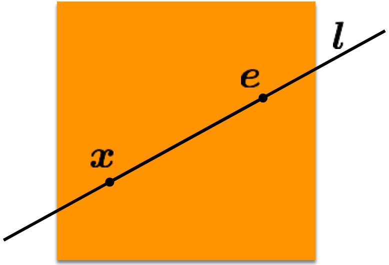


If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^{\top}\boldsymbol{l} = \ ?$$

Epipolar Line

$$ax + by + c = 0$$ in vector form $$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

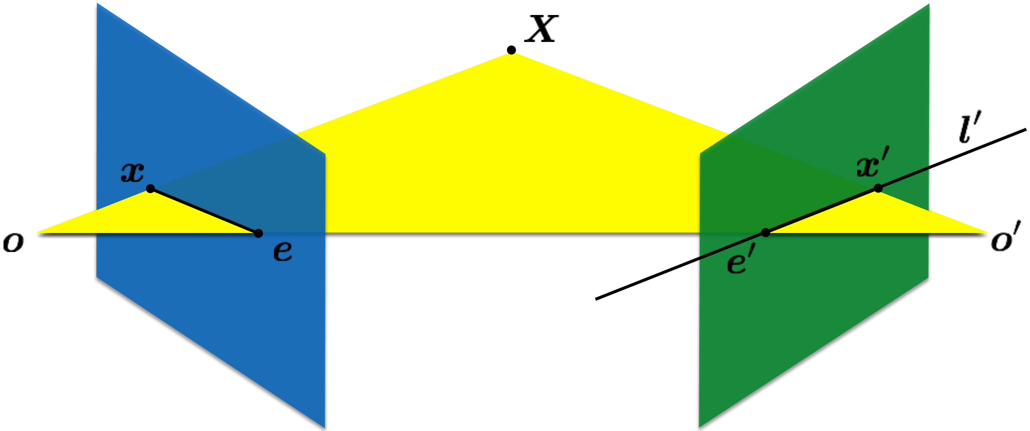


If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

So if $\boldsymbol{x}'^{\top} \boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boldsymbol{x}'^{\top} \mathbf{E} \boldsymbol{x} = \ ?$$



So if $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

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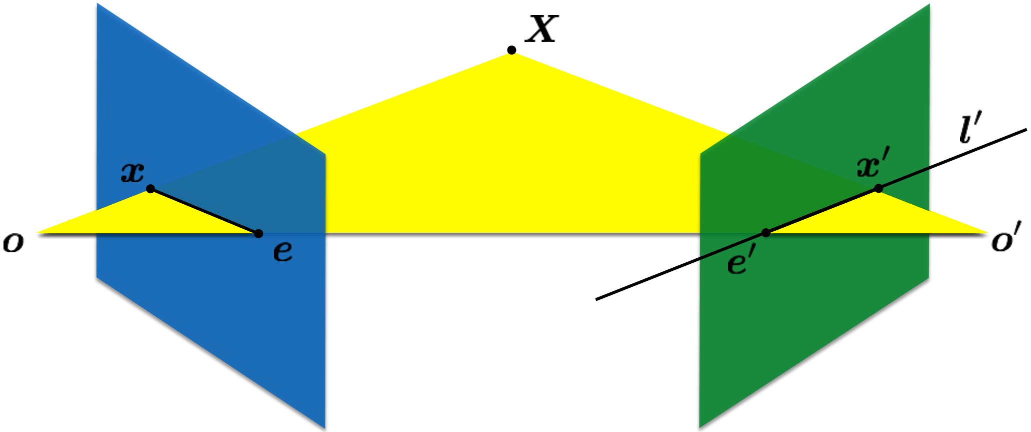


Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …

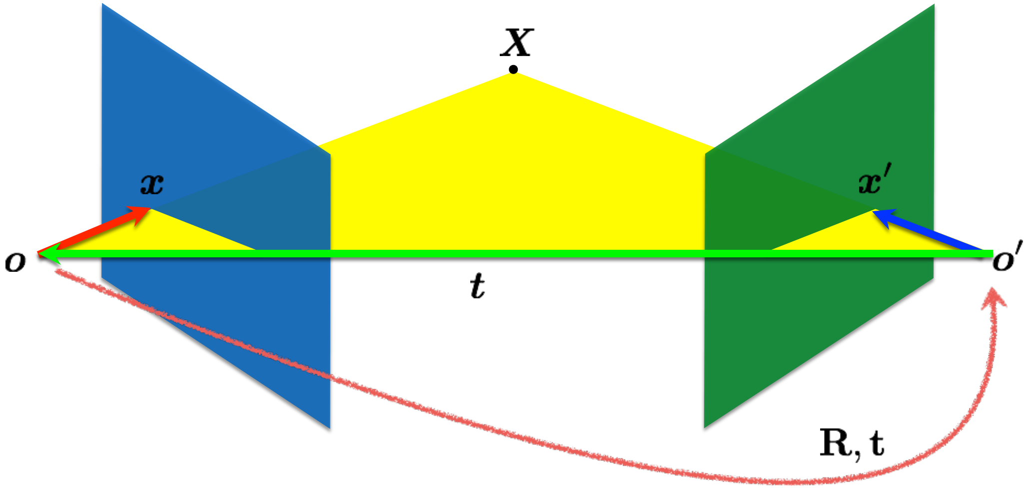
$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

Essential matrix maps a **point** to a **line**

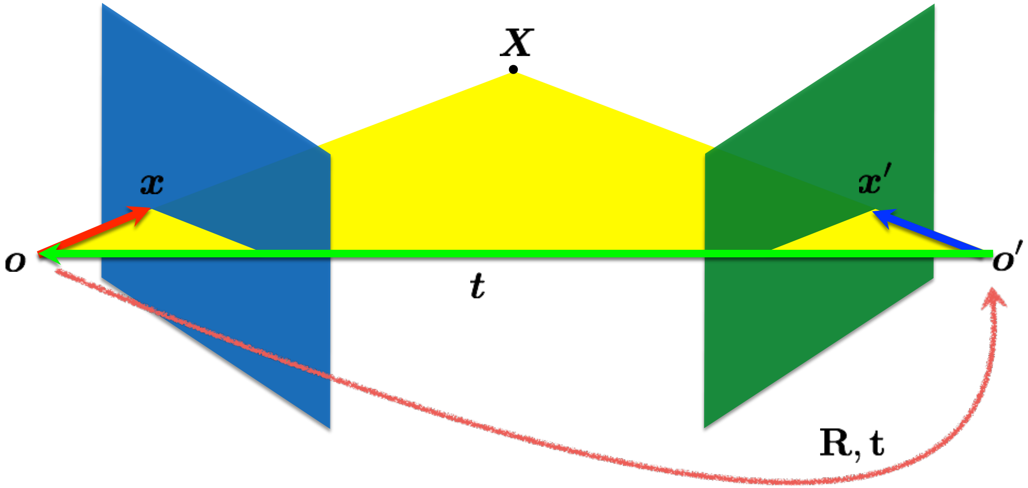
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

Homography maps a **point** to a **point**

Where does the essential matrix come from?

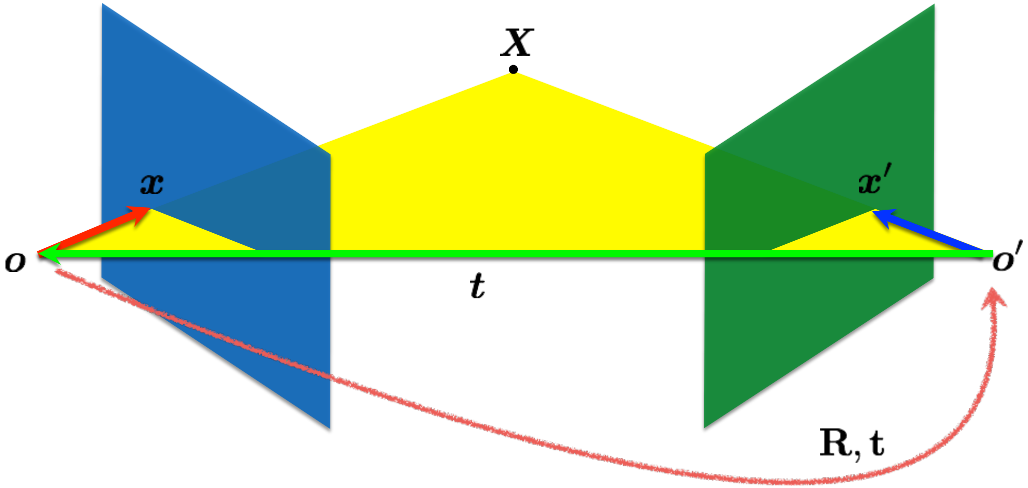


$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$



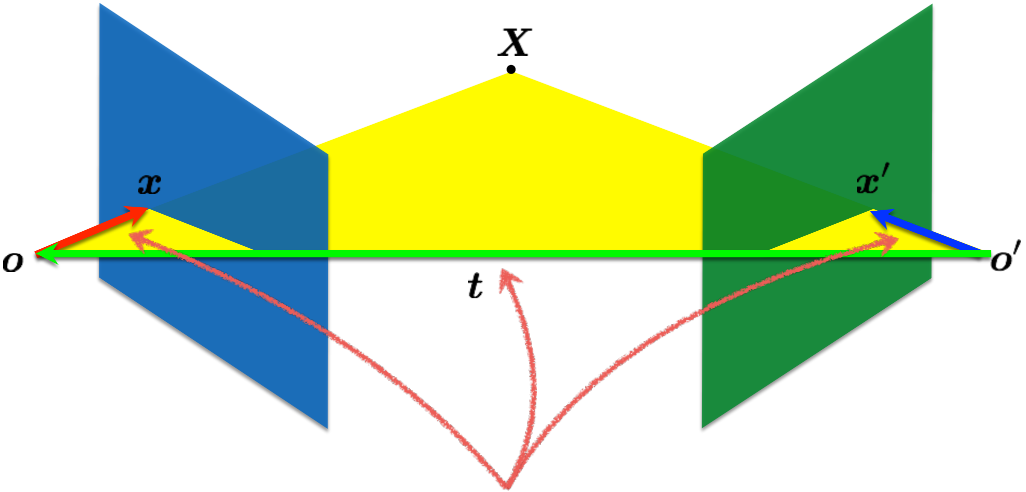
$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

Does this look familiar?



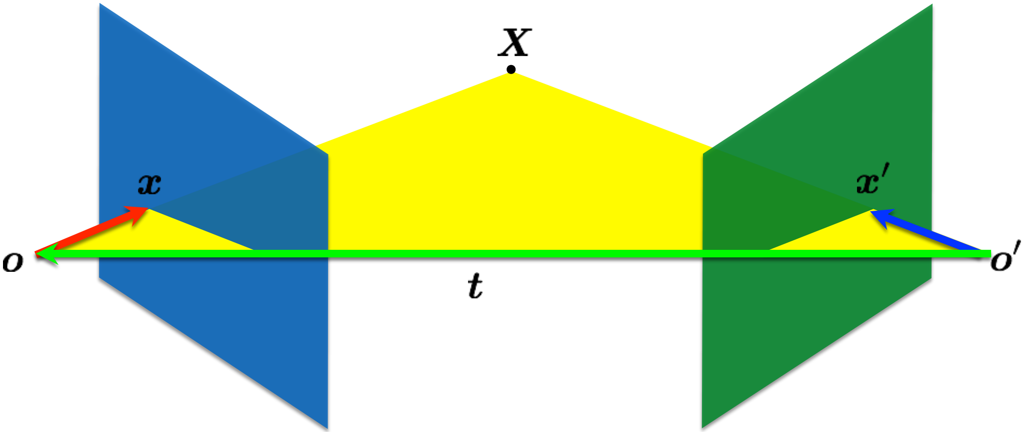
$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

Camera-camera transform just like **world-camera** transform



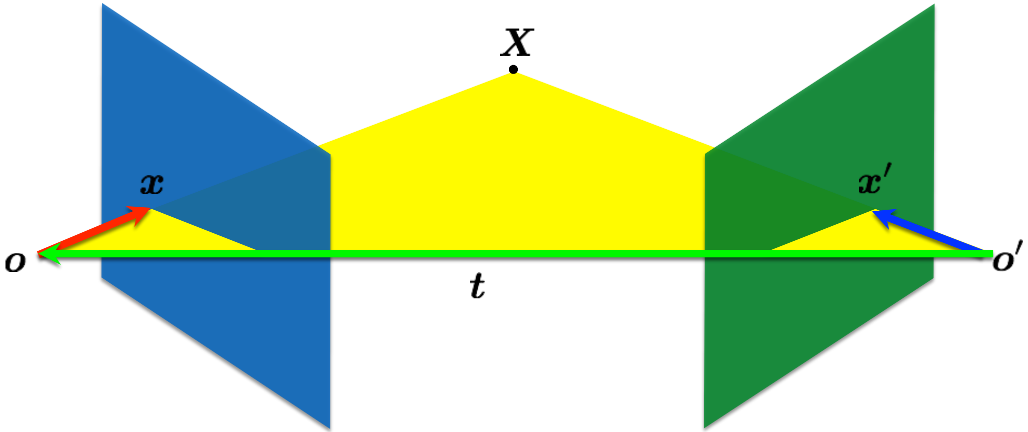
These three vectors are coplanar

$$\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$$



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top (\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$

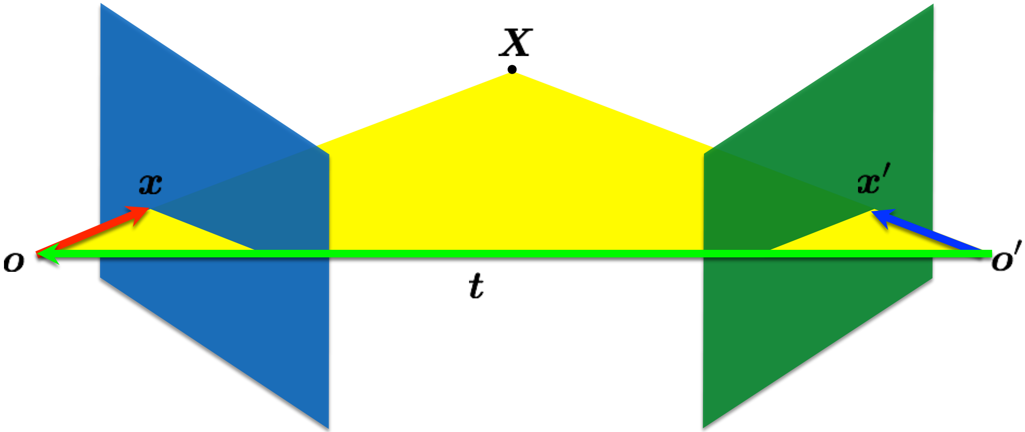


If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^{\top}(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

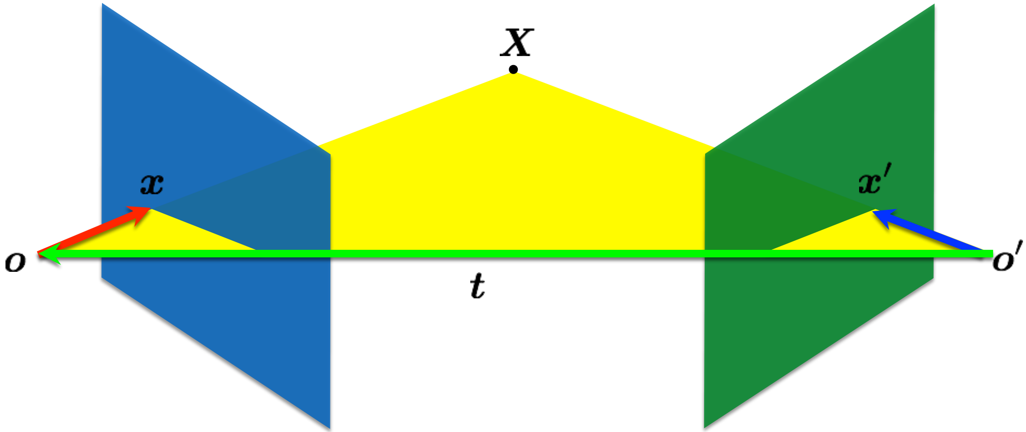
dot product of orthogonal vectors

cross-product: vector orthogonal to plane



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = \; ?$$



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

use skew-symmetric matrix to represent cross product

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

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$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

putting it together

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$$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0$$

putting it together

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coplanarity

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$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0}$$

Essential Matrix
[Longuet-Higgins 1981]

properties of the E matrix

Longuet-Higgins equation $\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

(2D points expressed in camera coordinate system)

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Epipolar lines

$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$ $\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$

$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$ $\boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$

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Epipoles $\boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0}$ $\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(2D points expressed in camera coordinate system)

Given a point in one image,
multiplying by the **essential matrix** will tell us
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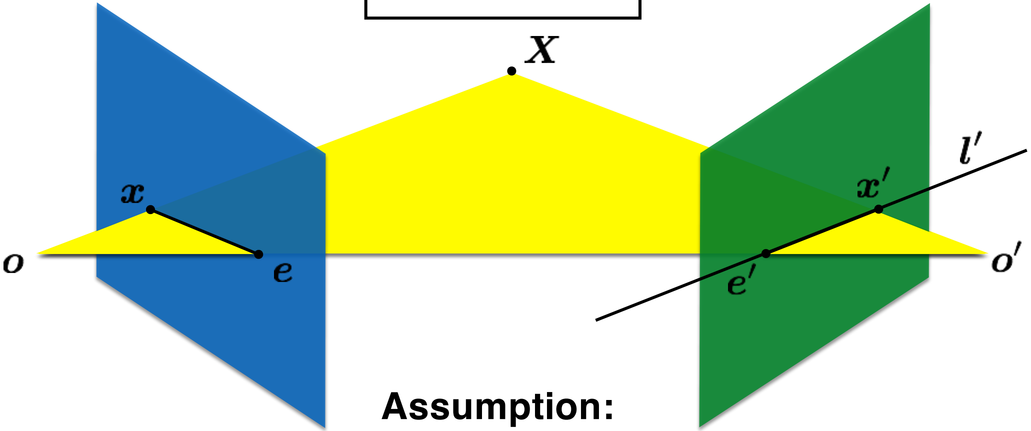
$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$



Assumption:
2D points expressed in camera coordinate system (i.e., intrinsic matrices are identities)

How do you generalize to non-identity intrinsic matrices?

The fundamental matrix

The

fundamental matrix

is a

generalization

of the

essential matrix,

where the assumption of

Identity matrices

is removed

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

The essential matrix operates on image points expressed in
2D coordinates expressed in the camera coordinate system

$$\hat{\boldsymbol{x}}' = \mathbf{K}'^{-1}\boldsymbol{x}'$$

$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

The essential matrix operates on image points expressed in **2D coordinates** expressed in the camera coordinate system

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Writing out the epipolar constraint in terms of image coordinates

$$\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

Same equation works in image coordinates!

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

it maps pixels to epipolar lines

properties of the F/E matrix

Longuet-Higgins equation	$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$	$\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$
	$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$	$\boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0}$	$\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(points in **image** coordinates)

Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

How would you solve for F?

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

The 8-point algorithm

Assume you have ***M*** matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

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S V D

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Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

Set up a homogeneous linear system with 9 unknowns

$$\boldsymbol{x}_m'^{\top} \mathbf{F} \boldsymbol{x}_m = 0$$

$$\begin{bmatrix} x_m' & y_m' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

How many equation do you get from one correspondence?

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

ONE correspondence gives you ONE equation

$$\begin{aligned} x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + \\ y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + \\ x'_m f_7 + y'_m f_8 + f_9 = 0 \end{aligned}$$

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

Set up a homogeneous linear system with 9 unknowns

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_Mx'_M & x_My'_M & x_M & y_Mx'_M & y_My'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = \mathbf{0}$$

How many equations do you need?

Each point pair (according to epipolar constraint)
contributes only one <u>scalar</u> equation

$$\boldsymbol{x}_m'^{\top} \mathbf{F} \boldsymbol{x}_m = 0$$

Note: This is different from the Homography estimation
where each point pair contributes 2 equations.

We need at least 8 points

Hence, the 8 point algorithm!

How do you solve a homogeneous linear system?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

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$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

Total Least Squares

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

How do you solve a homogeneous linear system?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

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minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

S V D !

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

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See Hartley-Zisserman for why we do this

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How do we do this?

S V D !

Enforcing rank constraints

Problem: Given a matrix F, find the matrix F' of rank k that is closest to F,

$$\min_{\substack{F' \\ \mathrm{rank}(F')=k}} \quad \|F - F'\|^2$$

Solution: Compute the singular value decomposition of F,

$$F = U\Sigma V^T$$

Form a matrix Σ' by replacing all but the k largest singular values in Σ with 0.

Then the problem solution is the matrix F' formed as,

$$F' = U\Sigma' V^T$$

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
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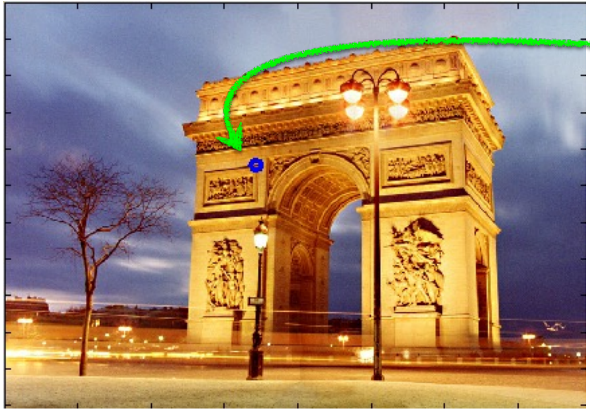
Example

epipolar lines

$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$



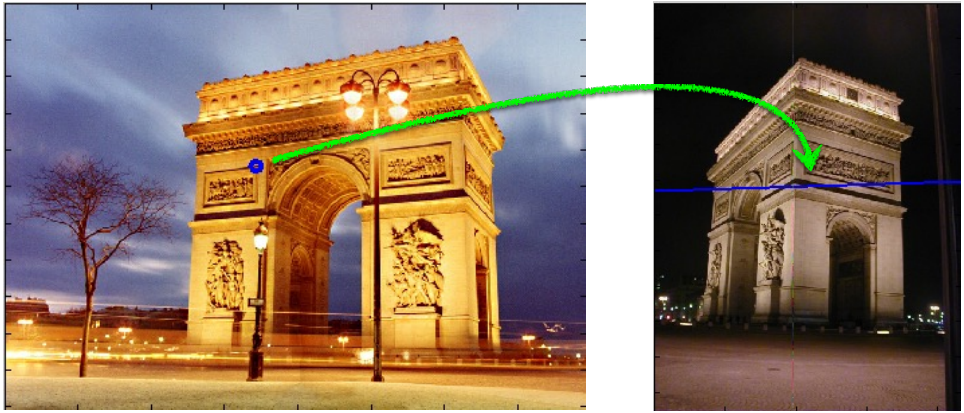
$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$



Where is the epipole?

How would you compute it?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

S V D !

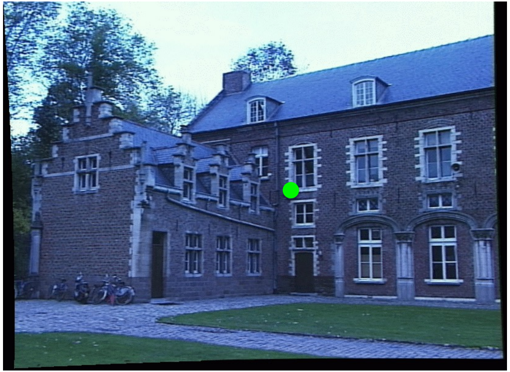
Revisiting triangulation

How would you reconstruct 3D points?

Left image

Right image

How would you reconstruct 3D points?

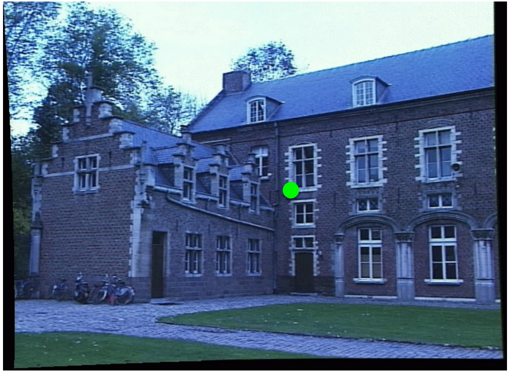


Left image

Right image

1. Select point in one image

How would you reconstruct 3D points?

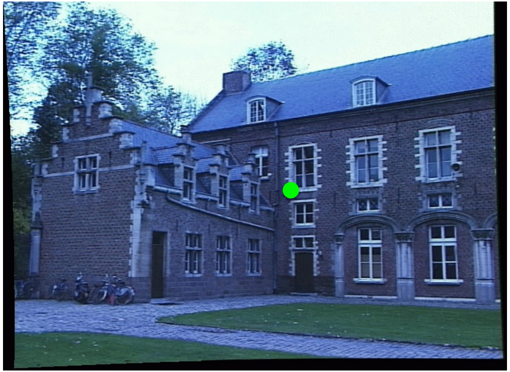


Left image

Right image

1. Select point in one image
2. Form epipolar line for that point in second image (how?)

How would you reconstruct 3D points?

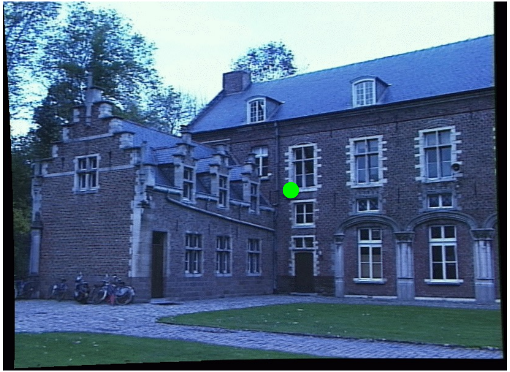


Left image

Right image

1. Select point in one image
2. Form epipolar line for that point in second image (how?)
3. Find matching point along line (how?)

How would you reconstruct 3D points?

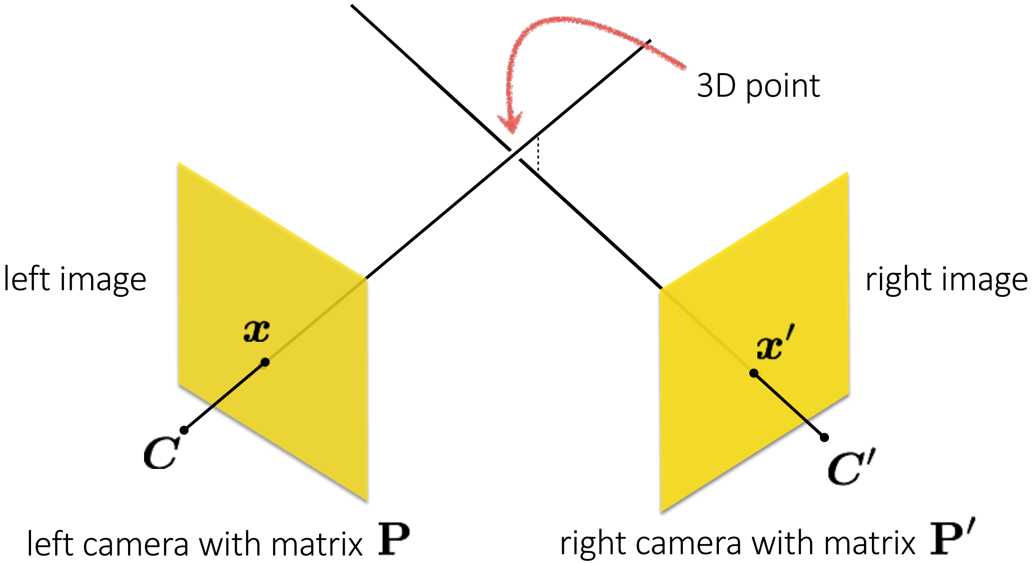


Left image

Right image

1. Select point in one image
2. Form epipolar line for that point in second image (how?)
3. Find matching point along line (how?)
4. Perform triangulation (how?)

Triangulation



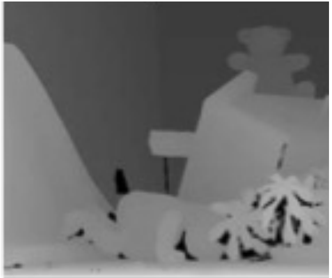
Stereo rectification

What’s different between these two images?

Objects that are close move more or less?

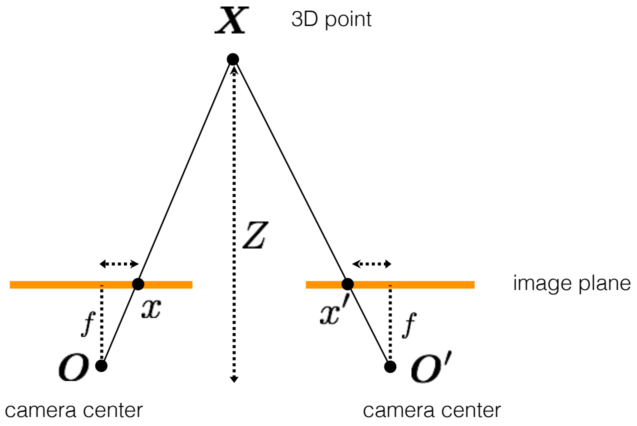
The amount of horizontal movement is inversely proportional to …

The amount of horizontal movement is
inversely proportional to …

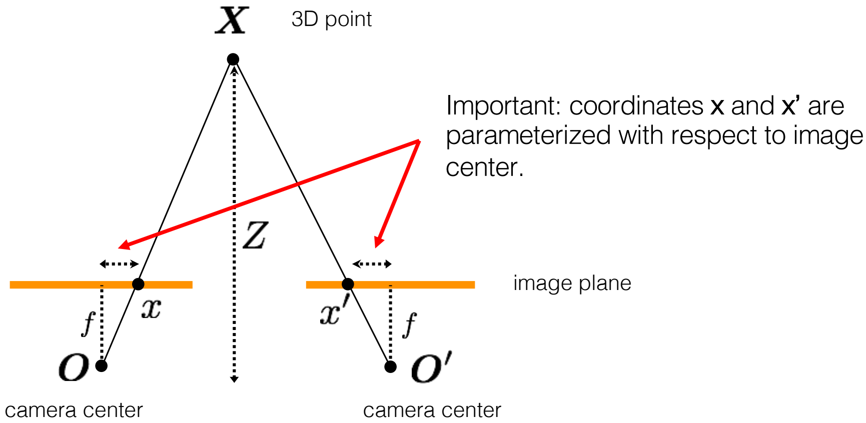
  

… the distance from the camera.

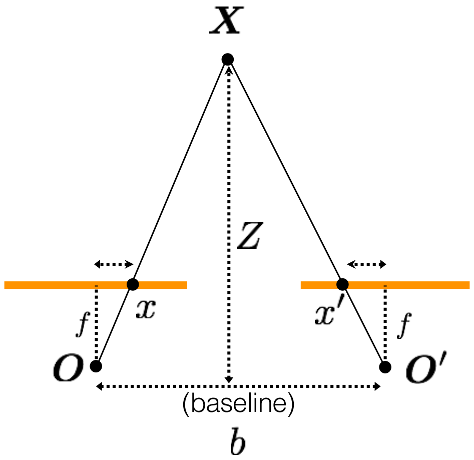
More formally…



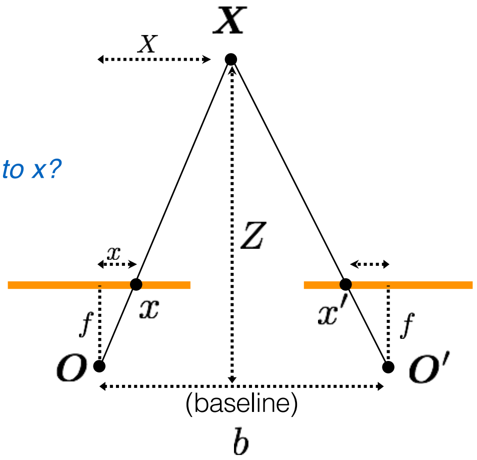
X
3D point
Z
image plane
x
x'
f
f
O
O'
camera center
camera center



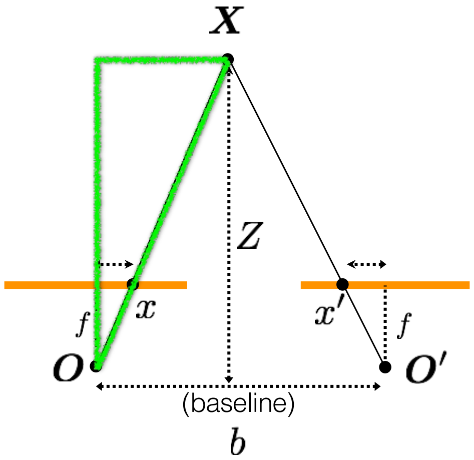
X
3D point
Important: coordinates x and x' are parameterized with respect to image center.
Z
image plane
x
x'
f
f
O
O'
camera center
camera center


$\boldsymbol{X}$
Z
x
x'
f
f
$\boldsymbol{O}$
$\boldsymbol{O'}$
(baseline)
b

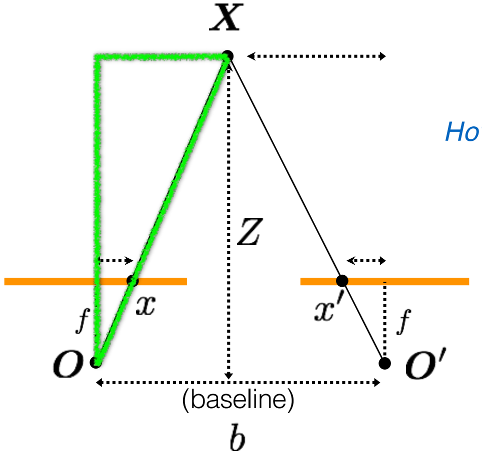
How is X related to x?



$$\frac{X}{Z} = \frac{x}{f}$$

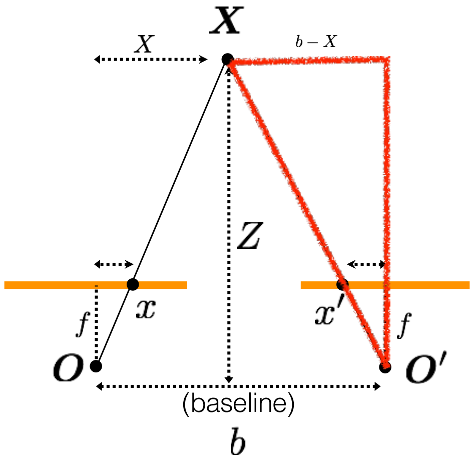


$$\frac{X}{Z} = \frac{x}{f}$$



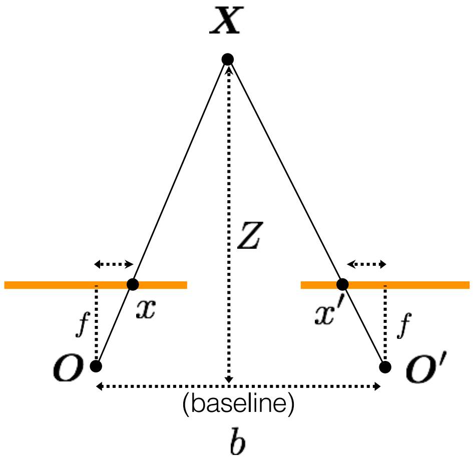
How is X related to x'?

$$\frac{X}{Z} = \frac{x}{f}$$



$$\frac{b - X}{Z} = \frac{-x'}{f}$$

$$\frac{X}{Z} = \frac{x}{f}$$



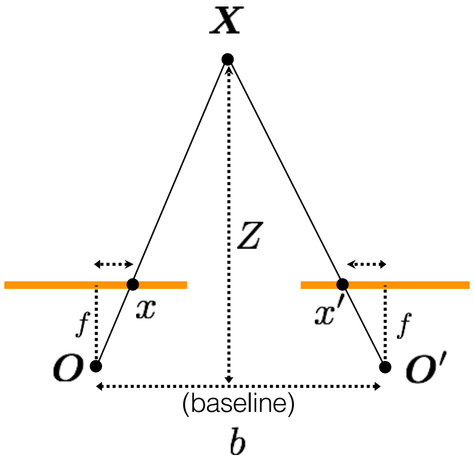
$$\frac{b - X}{Z} = \frac{-x'}{f}$$

Disparity

$$d = x - x'$$ (wrt to camera origin of image plane)

$$= \frac{bf}{Z}$$

$$\frac{X}{Z} = \frac{x}{f}$$



$$\frac{b - X}{Z} = \frac{-x'}{f}$$

Disparity

$$d = x - x'$$
$$= \frac{bf}{Z}$$

inversely proportional
to depth

Real-time stereo sensing

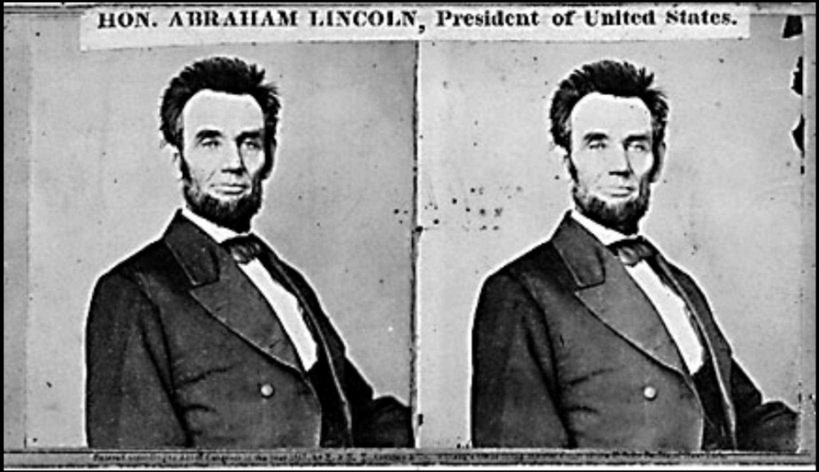
Nomad robot searches for meteorites in Antartica
http://www.frc.ri.cmu.edu/projects/meteorobot/index.html

Subaru
Eyesight system

Pre-collision
braking

What other vision system uses disparity for depth sensing?

Stereoscopes: A 19th Century Pastime


HON. ABRAHAM LINCOLN, President of United States.

Public Library, Stereoscopic Looking Room, Chicago, by Phillips, 1923

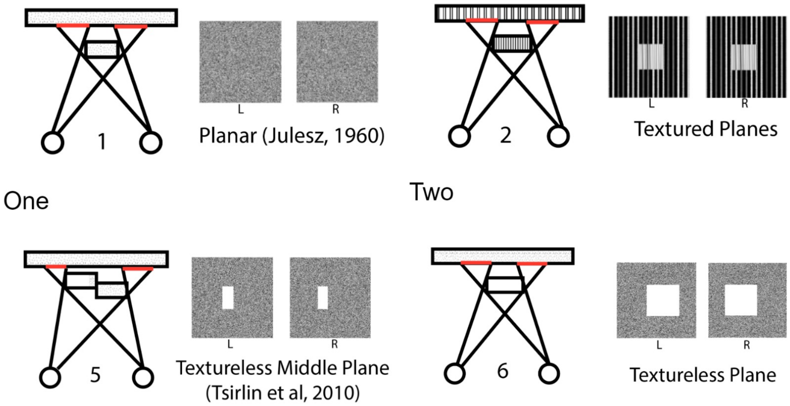
Teesta suspension bridge-Darjeeling, India

Mark Twain at Pool Table", no date, UCR Museum of Photography

This is how 3D movies work

Simple stereoscope

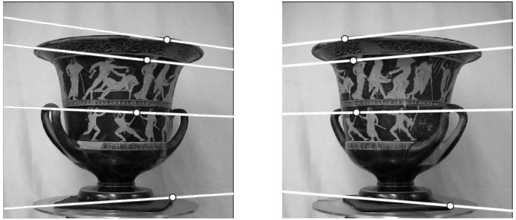
Google cardboard



Fun patterns: random dot stereograms

http://vision.seas.harvard.edu/stereo/

So can I compute depth using disparity from any two images of the same object?



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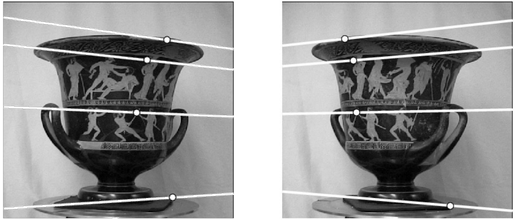
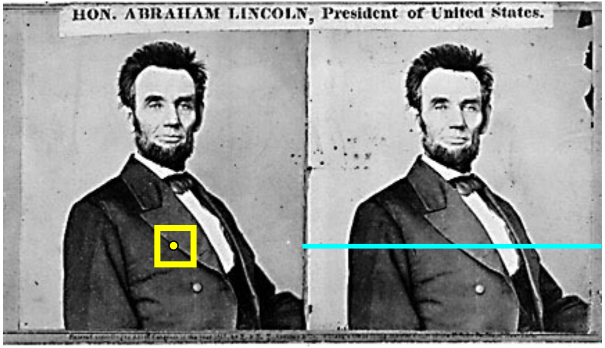
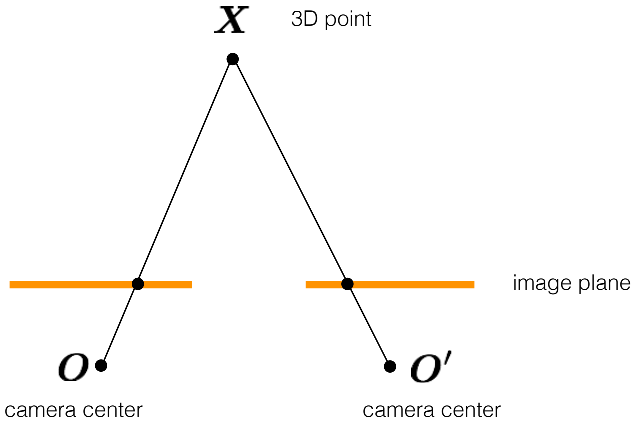


1. Need sufficient baseline

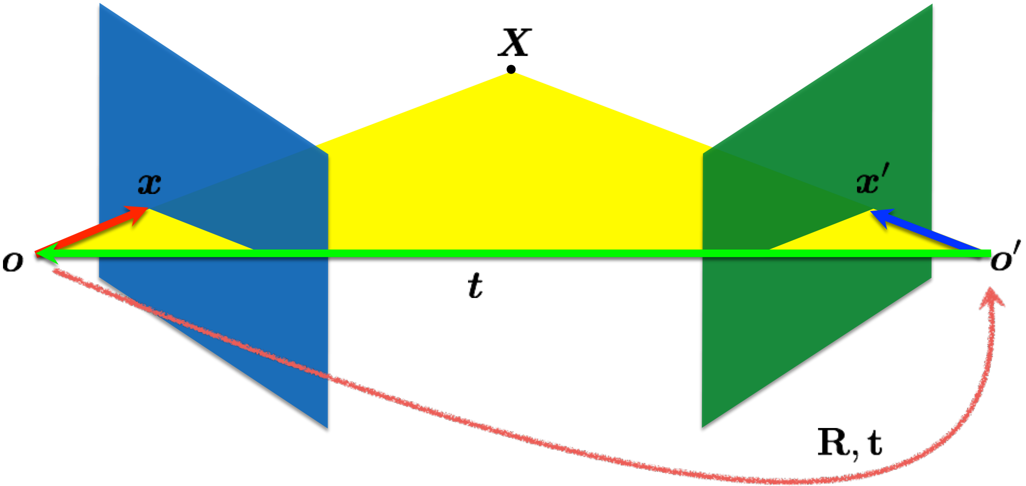
2. Images need to be 'rectified' first (make epipolar lines horizontal)



How can you make the epipolar lines horizontal?

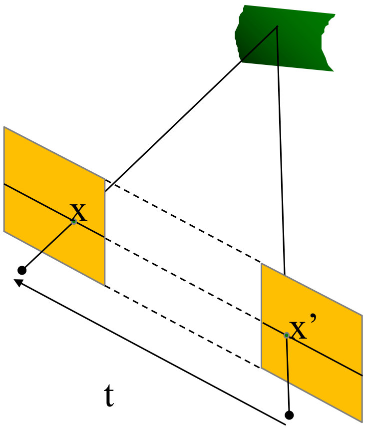


What's special about these two cameras?



$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

When are epipolar lines horizontal?



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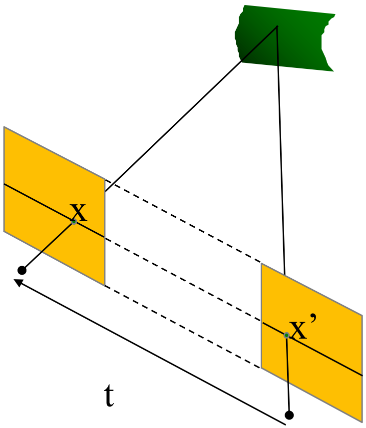
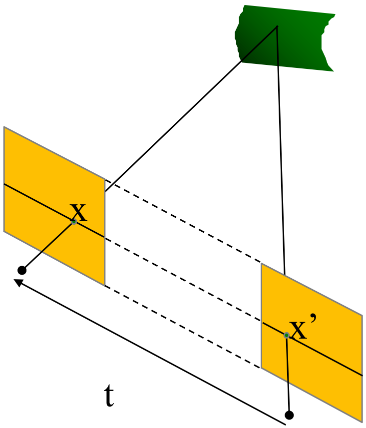


When this relationship holds:

$$\boxed{R = I \qquad\qquad t = (T, 0, 0)}$$

When are epipolar lines horizontal?



When this relationship holds:

$$\boxed{R = I \qquad t = (T, 0, 0)}$$

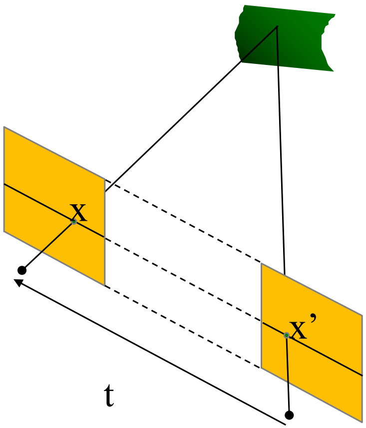
Let's try this out…

$$E = t \times R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

This always has to hold:

$$x^T E x' = 0$$

When are epipolar lines horizontal?



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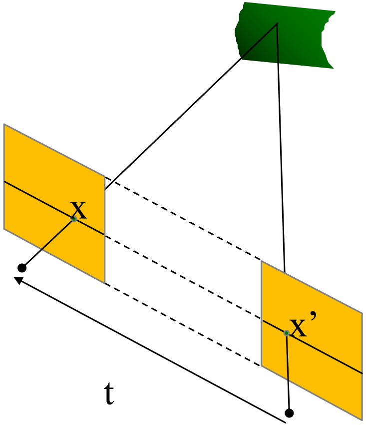
This always has to hold:

$$x^T E x' = 0$$

Write out the constraint

$$\begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0$$

When are epipolar lines horizontal?



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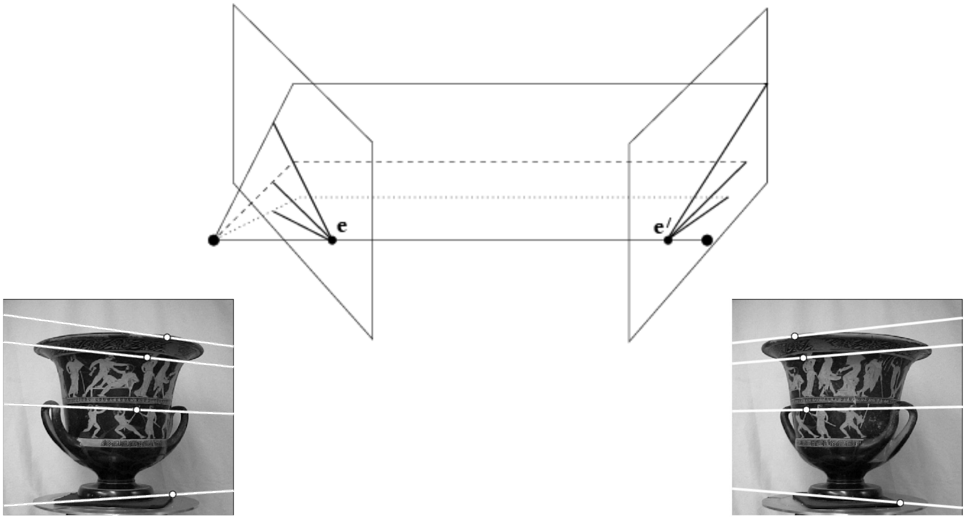
$$\begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0$$

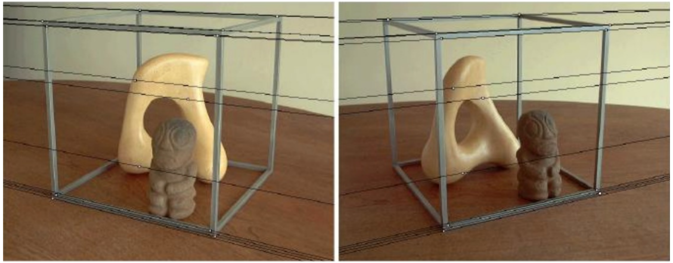
The image of a 3D point will always be on the same horizontal line

$$Tv = Tv'$$

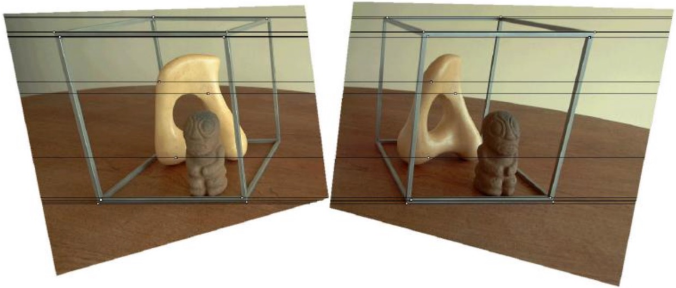
y coordinate is always the same!

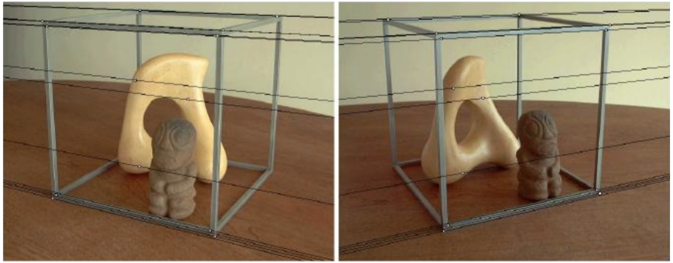


It’s hard to make the image planes exactly parallel

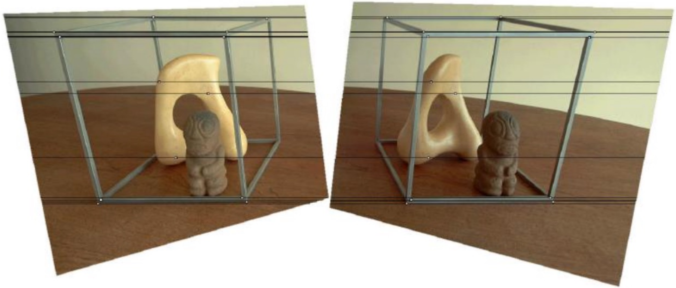


How can you make the epipolar lines horizontal?

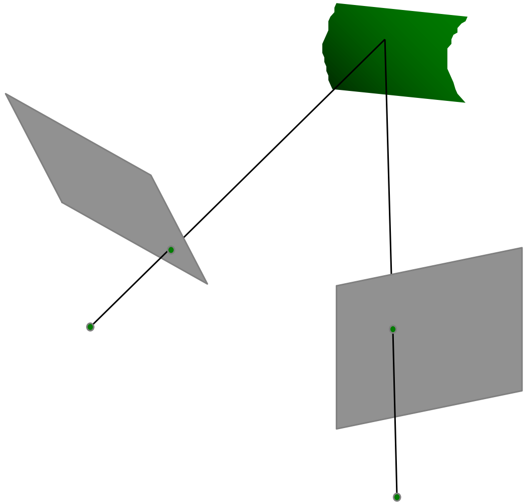




Use stereo rectification

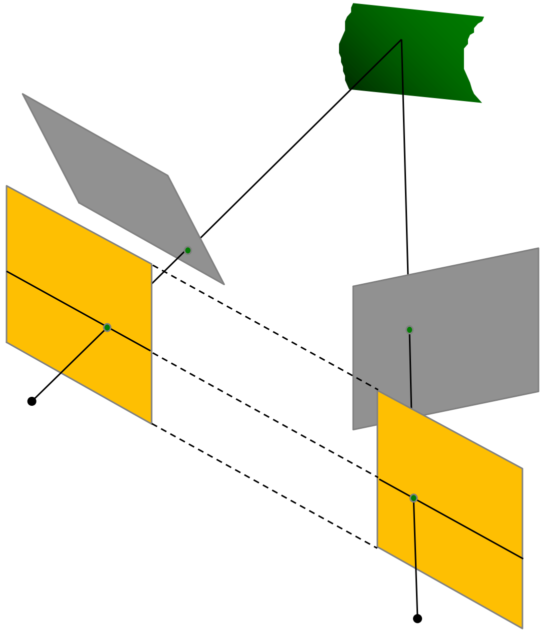


What is stereo rectification?



What is stereo rectification?

Reproject image planes onto a common plane parallel to the line between camera centers

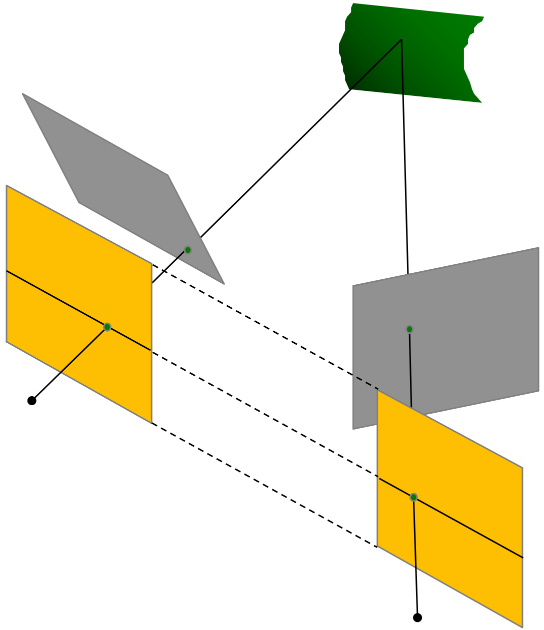


How can you do this?

What is stereo rectification?

Reproject image planes onto a common plane parallel to the line between camera centers

Need two homographies (3x3 transform), one for each input image reprojection



C. Loop and Z. Zhang. Computing Rectifying Homographies for Stereo Vision.Computer Vision and Pattern Recognition, 1999.

Stereo Rectification

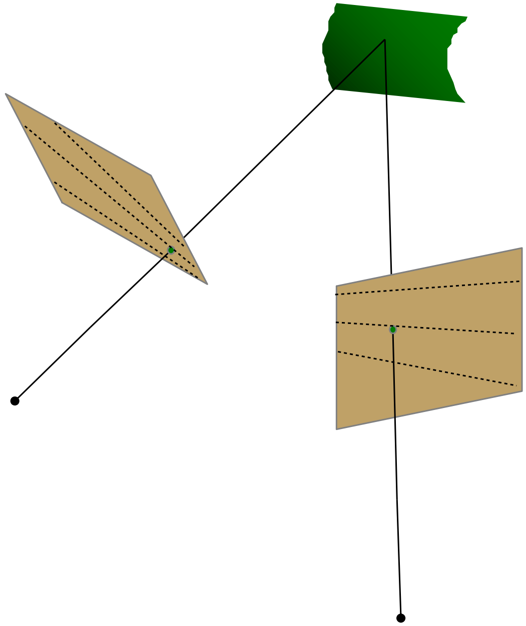
1. **Rotate** the right camera by **R**
   (aligns camera coordinate system orientation only)

2. Rotate (**rectify**) the left camera so that the epipole is at infinity

3. Rotate (**rectify**) the right camera so that the epipole is at infinity

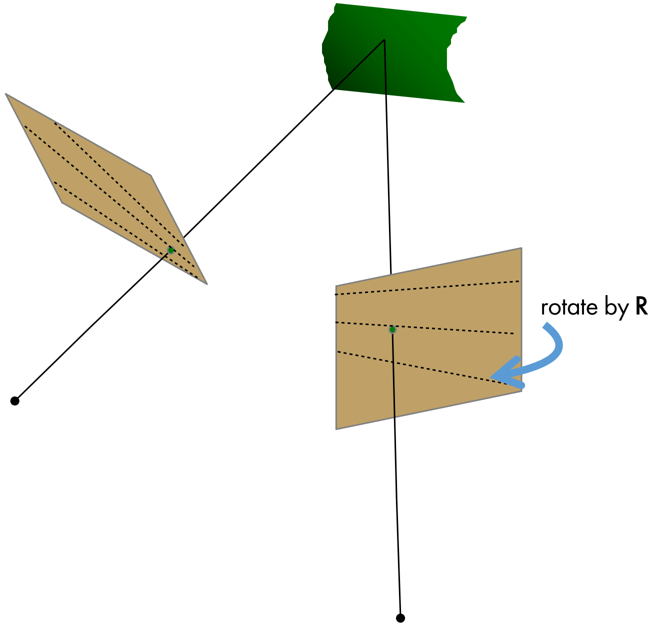
4. Adjust the **scale**

Stereo Rectification:



1. Compute **E** to get **R**
2. Rotate right image by **R**
3. Rotate both images by **R**$_{\mathbf{rect}}$
4. Scale both images by **H**

Stereo Rectification:



1. Compute **E** to get **R**
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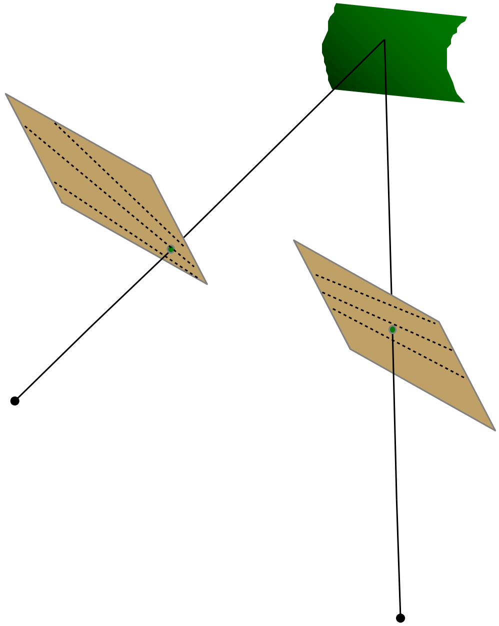
4. Scale both images by **H**

Stereo Rectification:



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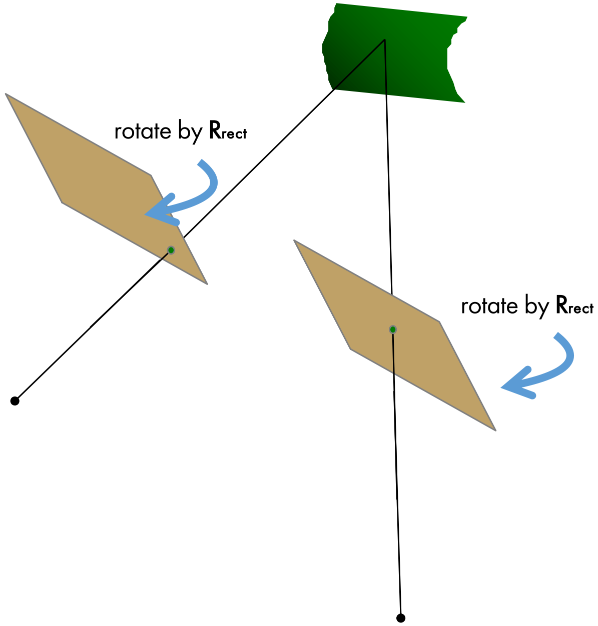

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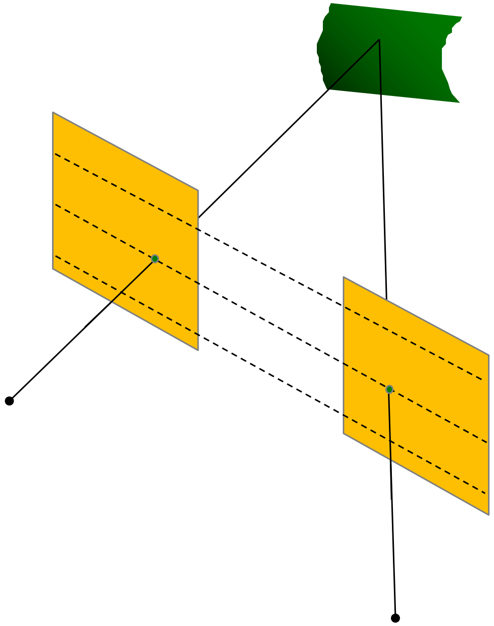

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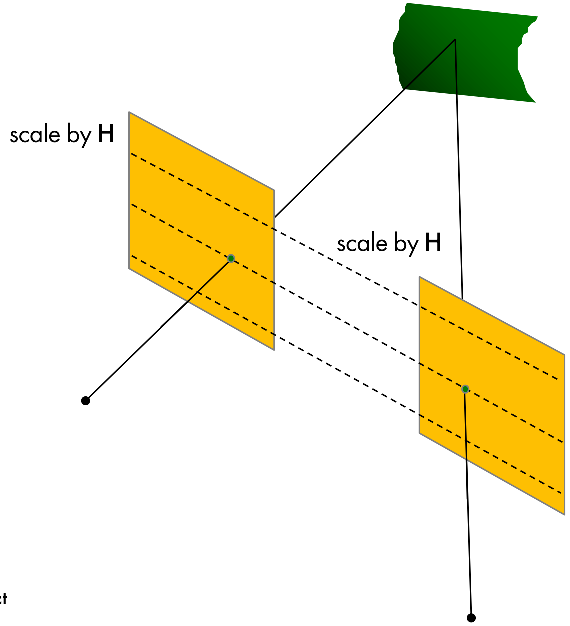

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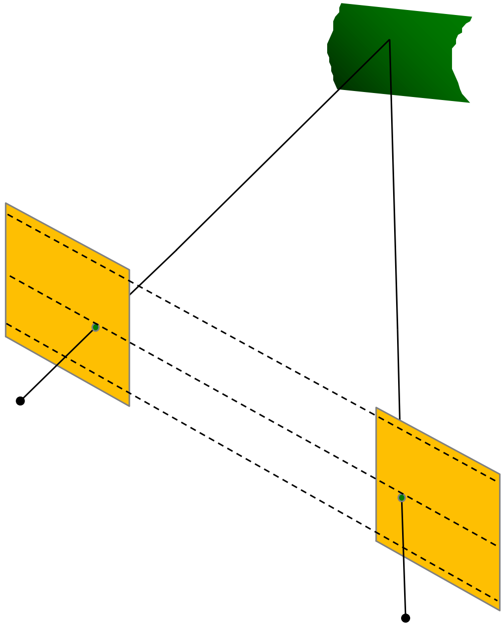

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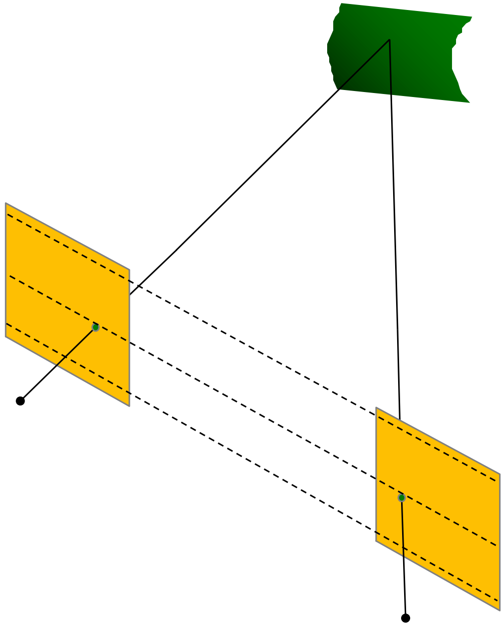

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4. Scale both images by **H**

Step 1: Compute $\mathbf{E}$ to get $\mathbf{R}$

SVD: $\mathbf{E} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ Let $\mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_\times \mathbf{R}$$

$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top$ $\mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top$ $\mathbf{T}_1 = U_3$ $\mathbf{T}_2 = -U_3$

two possible rotations

two possible translations

Step 1: Compute $\mathbf{E}$ to get $\mathbf{R}$

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_{\times}\mathbf{R}$$

$$\mathbf{T}_1 = U_3 \qquad \mathbf{T}_2 = -U_3$$

two possible translations

$$\mathbf{T}^T[\mathbf{T}]_{\times}\mathbf{R} = 0$$

so T must be the left nullspace of E

Step 1: Compute E to get R

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_\times \mathbf{R}$$

$$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top \quad \mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \qquad \mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

two possible rotations

Step 1: Compute E to get R

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_\times \mathbf{R}$$

$$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top \quad \mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \qquad \mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

two possible rotations

$$[\mathbf{t}]_\times \mathbf{R} = \mathbf{U}\,\mathbf{W}\,\mathbf{\Sigma}\,\mathbf{U}^T\mathbf{U}\,\mathbf{W}^{-1}\,\mathbf{V}^T = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^T = \mathbf{E}$$

Step 1: Compute E to get R

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_\times \mathbf{R}$$

$$\mathbf{R}_1 = \mathbf{UWV}^\top \quad \mathbf{R}_2 = \mathbf{UW}^\top\mathbf{V}^\top \qquad \mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

two possible rotations

$$[\mathbf{t}]_\times \, \mathbf{R} = \underbrace{\mathbf{U}\,\mathbf{W}\,\boldsymbol{\Sigma}\,\mathbf{U}^T}_{\text{this extracts } U_3}\mathbf{U}\,\mathbf{W}^{-1}\,\mathbf{V}^T = \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^T = \mathbf{E}$$

Step 1: Compute E to get R

We get FOUR solutions:

$$\mathbf{E} = [\mathbf{T}]_\times \mathbf{R}$$

$$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top \quad \mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \qquad \mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

two possible rotations

$$[\mathbf{t}]_\times \mathbf{R} = \mathbf{U}\,\mathbf{W}\,\mathbf{\Sigma}\,\mathbf{U}^T \underbrace{\mathbf{U}\,\mathbf{W}^{-1}\,\mathbf{V}^T} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^T = \mathbf{E}$$

all of these matrices are orthogonal, so the result is orthogonal—and a rotation matrix!

We get FOUR solutions:

$$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$
$$\mathbf{T}_1 = U_3$$

$$\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$
$$\mathbf{T}_2 = -U_3$$

$$\mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top$$
$$\mathbf{T}_2 = -U_3$$

$$\mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top$$
$$\mathbf{T}_1 = U_3$$

Which one do we choose?

Compute determinant of R, valid solution must be equal to 1

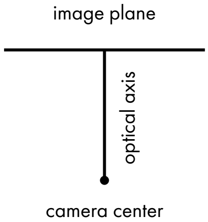
(note: det(R) = -1 means rotation and reflection)

Compute 3D point using triangulation, valid solution has positive Z value

(Note: negative Z means point is behind the camera)

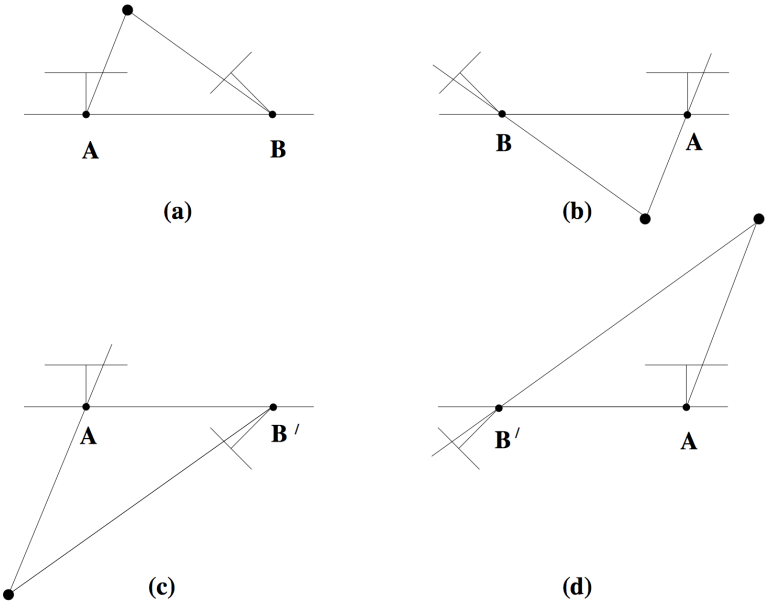
Let's visualize the four configurations...

Camera Icon

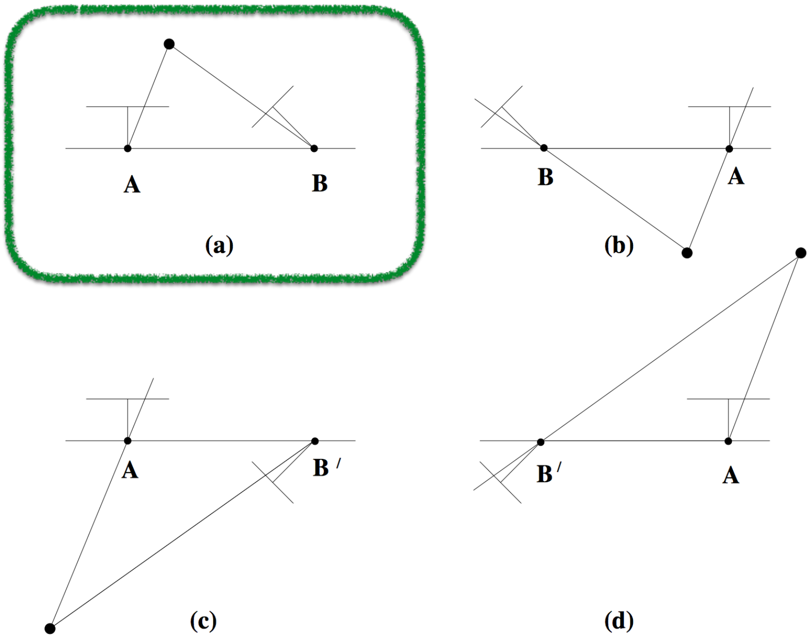


Find the configuration where the point is in front of both cameras

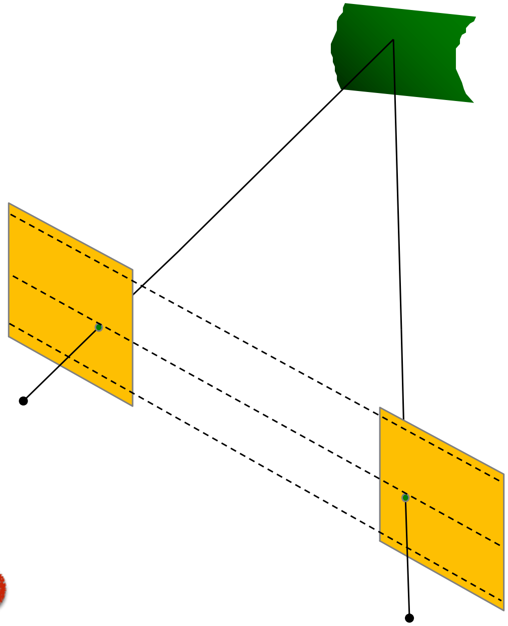
Find the configuration where the point is in front of both cameras



Find the configuration where the point is in front of both cameras



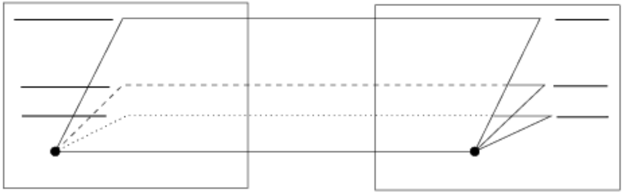
Stereo Rectification:

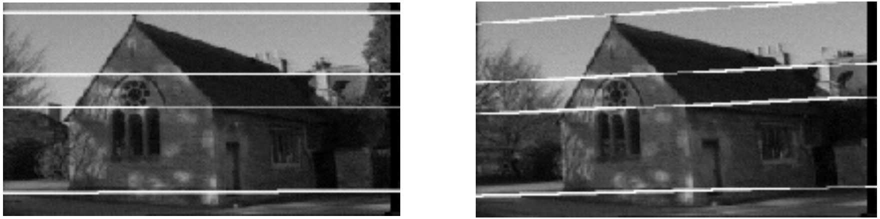


1. Compute E to get R
2. Rotate right image by R
3. Rotate both images by R_{rect}
4. Scale both images by H

When do epipolar lines become horizontal?

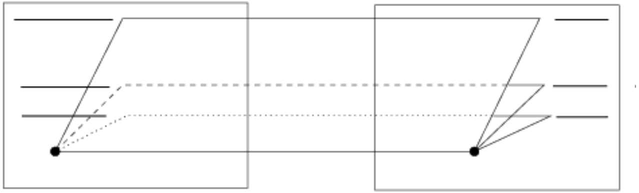
Parallel cameras

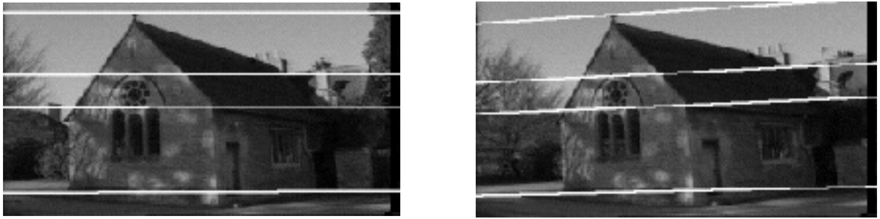




Where is the epipole?

Parallel cameras





epipole at infinity

Setting the epipole to infinity

(Building R_{rect} from E)

Let $$R_{\text{rect}} = \begin{bmatrix} \boldsymbol{r}_1^\top \\ \boldsymbol{r}_2^\top \\ \boldsymbol{r}_3^\top \end{bmatrix}$$

Given:

epipole e
(using SVD on E)
(translation from E)

$$\boldsymbol{r}_1 = \boldsymbol{e}_1 = \frac{T}{||T||}$$

epipole coincides with translation vector

$$\boldsymbol{r}_2 = \frac{1}{\sqrt{T_x^2 + T_y^2}} \begin{bmatrix} -T_y & T_x & 0 \end{bmatrix}$$

cross product of e and the direction vector of the optical axis [0, 0, 1]

$$\boldsymbol{r}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2$$

orthogonal vector

If $\boldsymbol{r}_1 = \boldsymbol{e}_1 = \dfrac{T}{||T||}$ and $\boldsymbol{r}_2$ $\boldsymbol{r}_3$ orthogonal

then $$R_{\text{rect}}\boldsymbol{e}_1 = \begin{bmatrix} \boldsymbol{r}_1^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_2^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_3^\top \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

If $\quad \boldsymbol{r}_1 = \boldsymbol{e}_1 = \dfrac{T}{||T||} \quad$ and $\quad \boldsymbol{r}_2 \quad \boldsymbol{r}_3 \quad$ orthogonal

then $\quad R_{\text{rect}}\boldsymbol{e}_1 = \begin{bmatrix} \boldsymbol{r}_1^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_2^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_3^\top \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Where is this point located on the image plane?

If $\boldsymbol{r}_1 = \boldsymbol{e}_1 = \frac{T}{||T||}$ and $\boldsymbol{r}_2$ $\boldsymbol{r}_3$ orthogonal

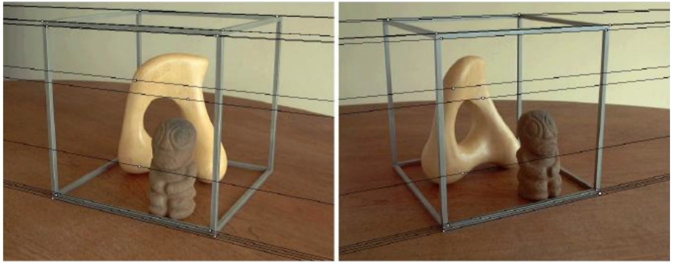
then $R_{\text{rect}}\boldsymbol{e}_1 = \begin{bmatrix} \boldsymbol{r}_1^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_2^\top \boldsymbol{e}_1 \\ \boldsymbol{r}_3^\top \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Where is this point located on the image plane?

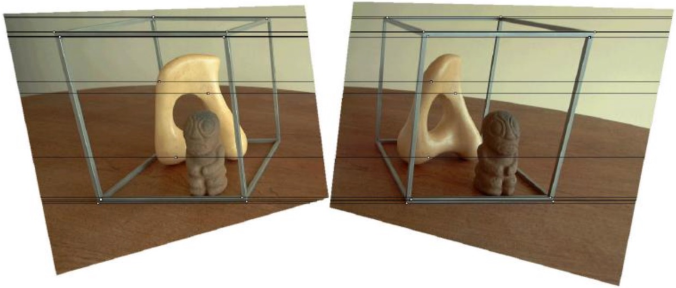
At x-infinity

Stereo Rectification Algorithm

1. Estimate $\mathbf{E}$ using the 8 point algorithm
2. Estimate the epipole $\mathbf{e}$ (solve $\mathbf{Ee}=0$)
3. Build $\mathbf{R}_{rect}$ from $\mathbf{e}$
4. Decompose $\mathbf{E}$ into $\mathbf{R}$ and $\mathbf{T}$
5. Set $\mathbf{R}_1=\mathbf{R}_{rect}$ and $\mathbf{R}_2 = \mathbf{R}_{rect}\mathbf{R}$
6. Rotate each left camera point $\mathbf{x}' \sim \mathbf{Hx}$ where $\mathbf{H} = \mathbf{KR}_1$
   *You may need to alter the focal length (inside $\mathbf{K}$) to keep points within the original image size
7. Repeat 6 and 7 for right camera points using $\mathbf{R}_2$



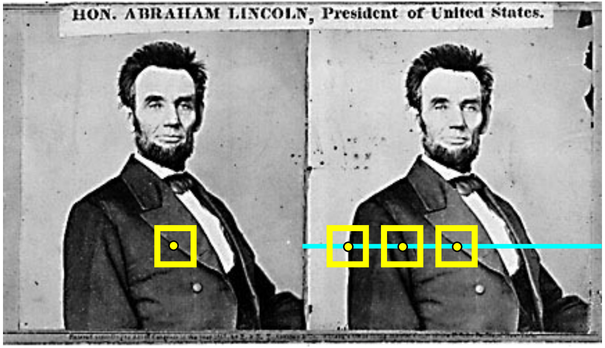
What can we do after rectification?



Stereo matching

Stereo matching

Depth Estimation via Stereo Matching



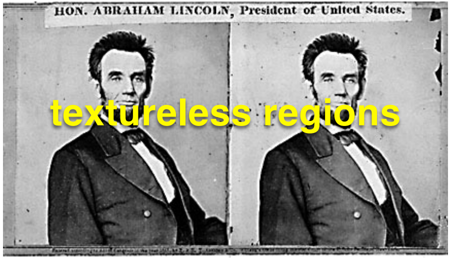
1. Rectify images
   (make epipolar lines horizontal)
2. For each $\boxed{\text{pixel}}$
   a. Find epipolar line
   b. Scan line for best match
   c. Compute depth from disparity

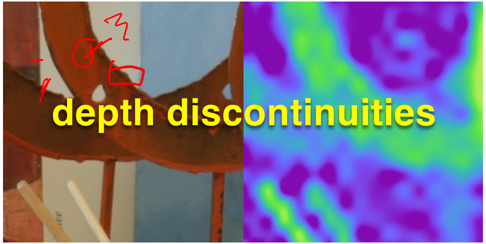
$$Z = \frac{bf}{d}$$

How would you do this?

When are correspondences difficult?

When are correspondences difficult?

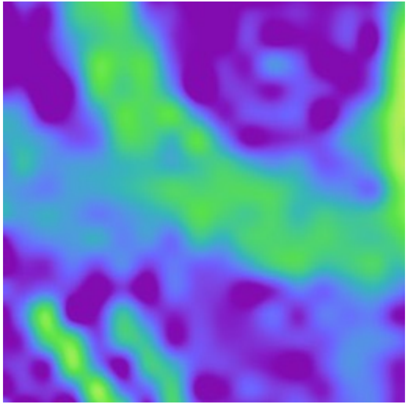




Depth discontinuities

What is the problem here?

One of two input images



Depth from disparity

Groundtruth depth

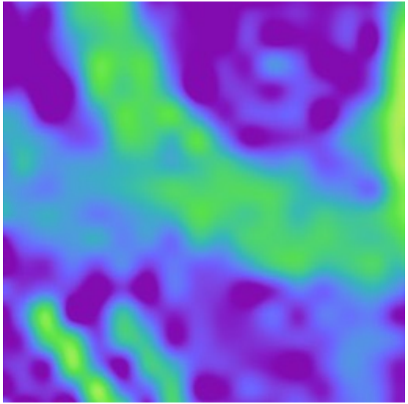
Depth discontinuities

What is the problem here?

- (Patch-wise) stereo matching blurs along the edges.

How can we fix this?

One of two input images



Depth from disparity

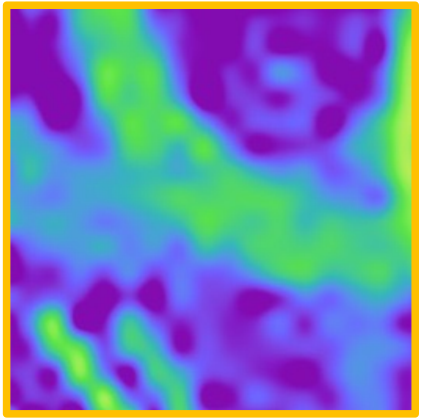
Groundtruth depth

Edge-aware depth denoising

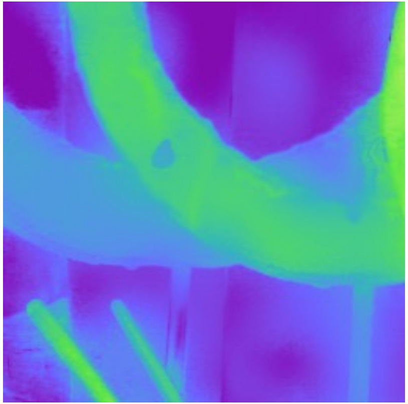
$$A_{p(col)} = \frac{1}{k(p(col))} \sum_{p' \in \Omega} g_d(|p - p'|)\, g_r\left(F_{p(col)} - F_{p'(col)}\right) A_{p'(col)}$$

Use joint bilateral filtering, with the input image as guide.

One of two input images



Depth from disparity

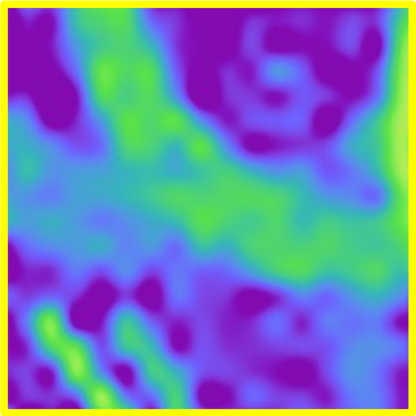


Guided filtering

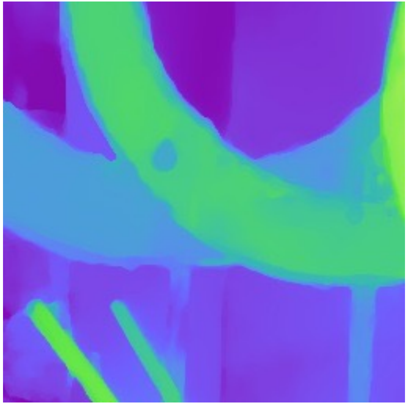
Fast bilateral solver

Possible to *combine* edge-enforcement and matching in a single optimization problem, instead of just filtering in post-processing.

One of two input images

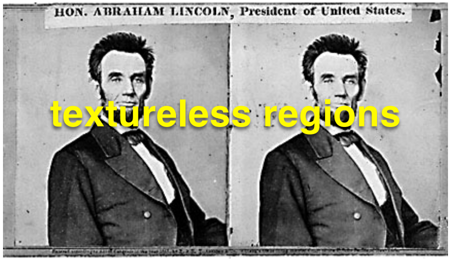


Depth from disparity



Bilateral stereo matching

When are correspondences difficult?

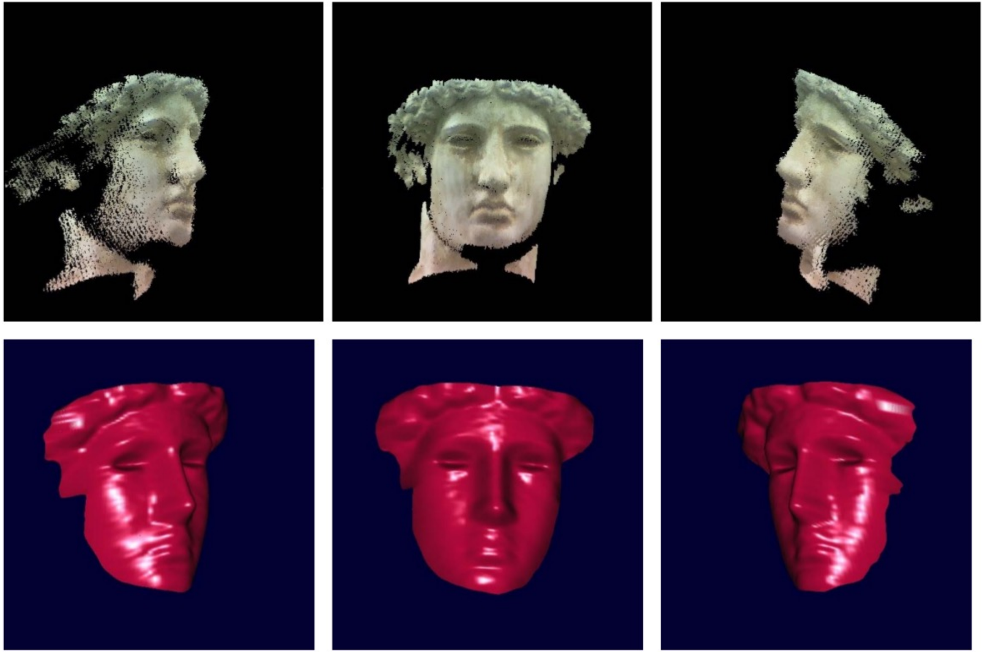


Structured light

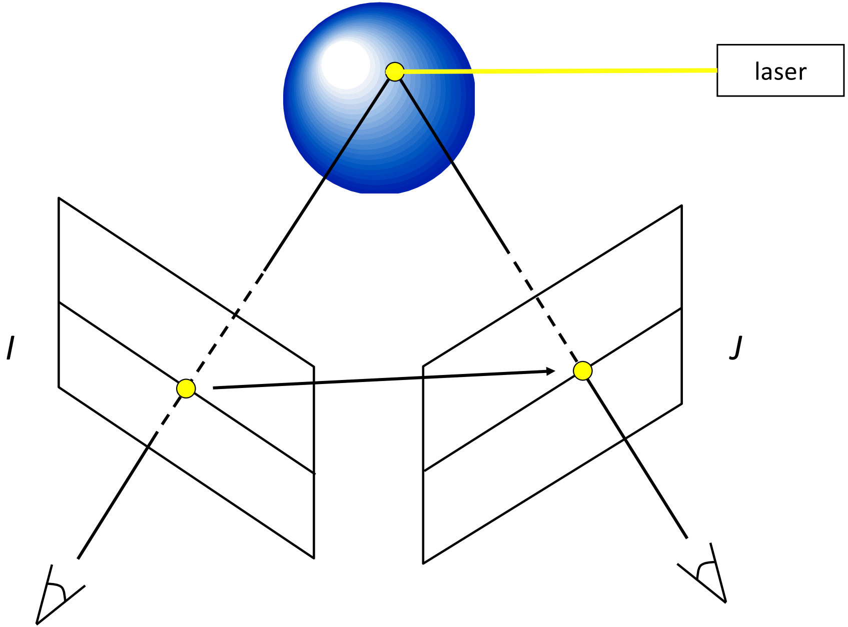
Use controlled (“structured”) light to make correspondences easier

Disparity between laser points on the same scanline in the images determines the 3-D coordinates of the laser point on object

Use controlled ("structured") light to make correspondences easier

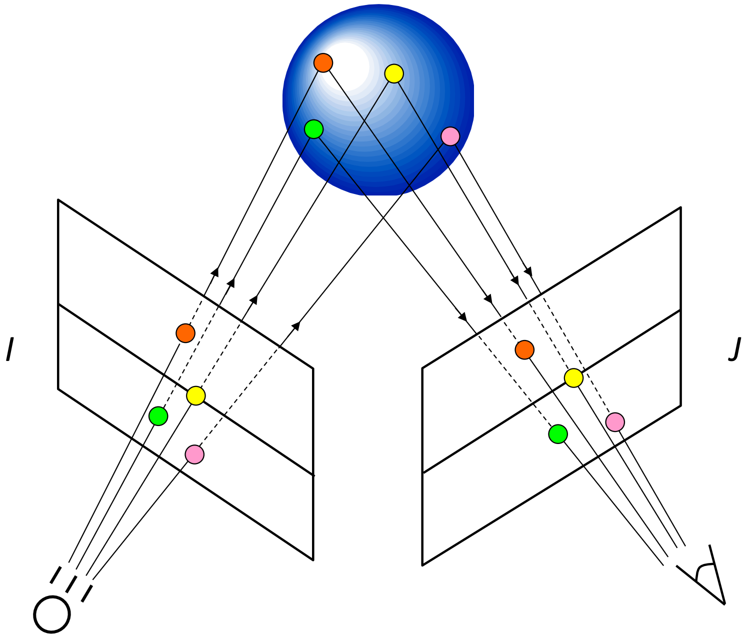


Structured light and two cameras

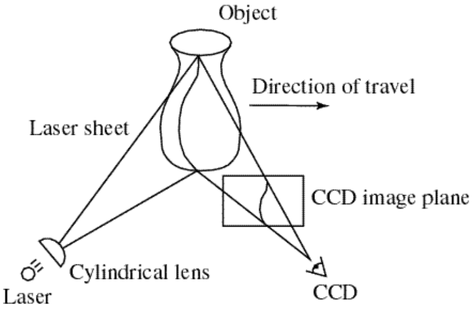


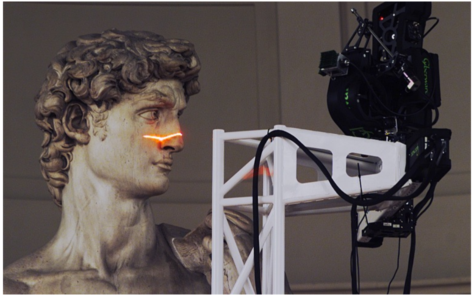
Structured light and one camera

Projector acts like
“reverse” camera



Example: Laser scanner

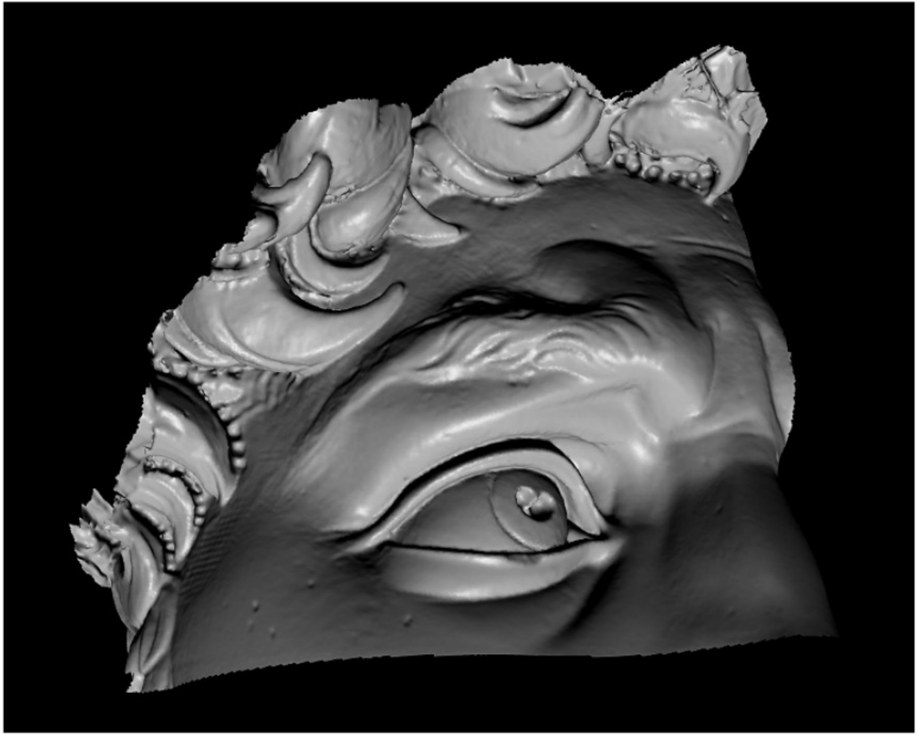




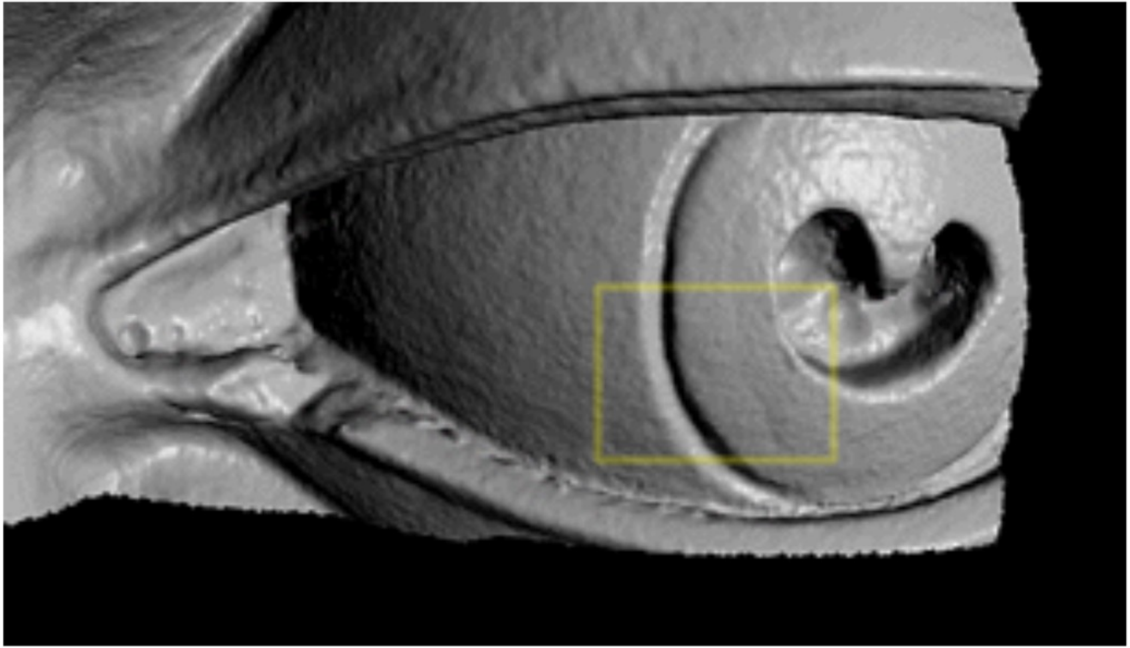
Digital Michelangelo Project
http://graphics.stanford.edu/projects/mich/

The Digital Michelangelo Project, Levoy et al.

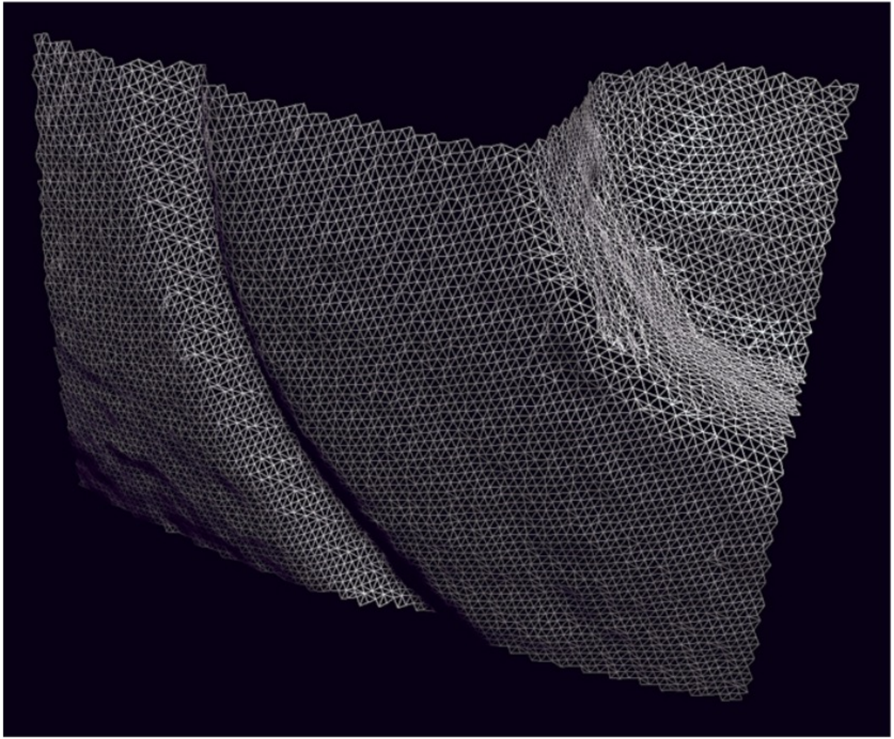
The Digital Michelangelo Project, Levoy et al.



The Digital Michelangelo Project, Levoy et al.



The Digital Michelangelo Project, Levoy et al.



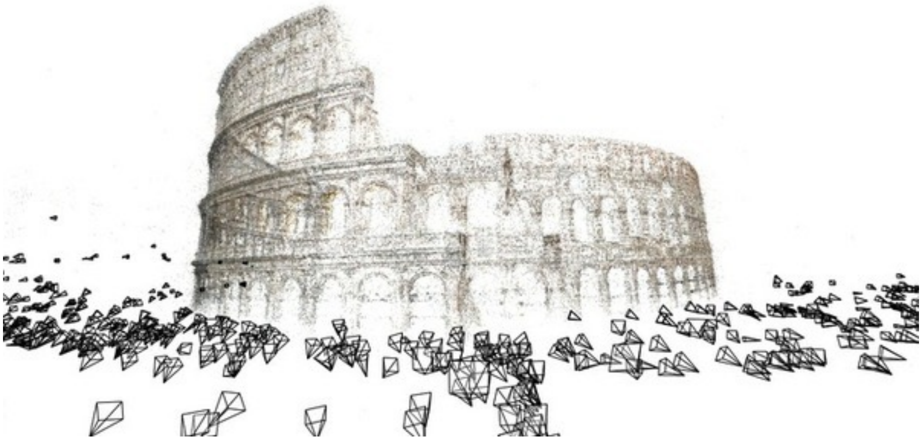
The Digital Michelangelo Project, Levoy et al.

Overview

- Triangulation
- Epipolar geometry
- Essential matrix & Fundamental matrix
- 8-point algorithm
- Stereo rectification
- Stereo matching & structured light

Next Time

Structure from Motion & NeRFs!



References

Basic reading:

- Szeliski textbook, Section 8.1 (not 8.1.1-8.1.3), Chapter 11, Section 12.2.
- Hartley and Zisserman, Section 11.12.

References

Basic reading:

- Szeliski textbook, Sections 7.1, 11.1, 12.1.
- Boles et al., “Epipolar-plane image analysis: An approach to determining structure from motion,” IJCV 1987.
  - This classical paper introduces EPIs, and discusses how they can be used to infer depth.
- Lanman and Taubin, “Build Your Own 3D Scanner: Optical Triangulation for Beginners,” SIGGRAPH course 2009.
  - This very comprehensive course has everything you need to know about 3D scanning using structured light, including details on how to build your own.
- Bouguet and Perona, “3D Photography Using Shadows in Dual-Space Geometry,” IJCV 1999.
  - This paper introduces the idea of using shadows to do structured light 3D scanning, and shows an implementation using just a camera, desk lamp, and a stick.

Additional reading:

- Gupta et al., “A Practical Approach to 3D Scanning in the Presence of Interreflections, Subsurface Scattering and Defocus,” IJCV 2013.
  - This paper has a very detailed treatment of standard patterns used for structured light, problems arising due to global illumination, and robust patterns for dealing with these patterns.
- Barron et al., “Fast bilateral-space stereo for synthetic defocus,” CVPR 2015.
- Barron and Poole, “The fast bilateral solver,” ECCV 2016.
  - The above two papers show how to combine edge-aware filtering (and bilateral filtering in particular) with disparity matching for robust stereo. The first paper also shows how the resulting depth maps can be used to create synthetic defocus blur.
- Wanner and Goldluecke, “Globally Consistent Depth Labeling of 4D Light Fields,” CVPR 2012.
- Kim et al., “Scene reconstruction from high spatio-angular resolution light fields,” SIGGRAPH 2013.
  - These two papers show detailed systems for using EPIs to extract depth.
- Levin et al., “Understanding camera trade-offs through a Bayesian analysis of light field projections,” ECCV 2008.
  - This paper uses EPIs to show how different types of imaging systems (pinhole cameras, plenoptic cameras, stereo pairs, lens-based systems, and so on) relate to each other, and analyze their pros and cons for 3D imaging.