Projective Transforms & RANSAC

CSC420
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Slide credit: Babak Taati ← Ahmed Ashraf ← Sanja Fidler, Yannis Gkioulekas

Christopher Moloney
Logistics

- A3 due on Friday!
Hi all,

For the Laplacian of Gaussian filter in A3 Question 1, you should be sure to use the normalized version of this filter so that the response at different scales is consistent. See the below image, which illustrates the difference. More information can be found on these slides (http://cubs.buffalo.edu/~inwogu/teaching/Coursepage573_fa14/lectures/12LocalFeatures_contd.pdf).

More specifically, you should use this equation, but add a scale factor of $\sigma^2$ to compute the normalized Laplacian of Gaussian filter (this results in a factor of $1/\sigma^2$ in the denominator at the front of the expression).

- Laplacian of Gaussian: We mentioned it for edge detection

$$\nabla^2_{\sigma}(x, y, \sigma) = -\frac{1}{\pi\sigma^4} \left(1 - \frac{x^2 + y^2}{2\sigma^2}\right) \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$
Overview

• Motivation: image matching, panoramas
• What is a homography?
• computing homographies
• Random sample consensus (RANSAC)
Recap from last time

• Now we know how to extract scale and rotation invariant features
• We even know how to match features across images
• Can we use this to find Waldo in an even more sneaky scenario?

Someone takes a (weird) picture of him!
Find My DVD!

• More interesting: If we have DVD covers (e.g., from Amazon), can we match them to DVDs in real scenes?
What Transformation Happened To My DVD?

- Rectangle goes to a parallelogram
Affine Transformsations

Affine transformations are combinations of
• Linear transformations, and translations

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
a & b & e \\
c & d & f
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

• Properties of affine transformations:
  • Origin does not necessarily map to origin
  • Lines map to lines
  • Parallel lines remain parallel
  • Ratios are preserved
  • Closed under composition

  • Rectangles go to parallelograms

[Source: N. Snavely, slide credit: R. Urtasun]
What Transformation Happened To My DVD?

• Is this an affine transformation?
What Transformation Happened To My DVD?

- Actually, a rectangle, maps to quadrilateral
2D Image Transformations

These transformations are a nested set of groups
Closed under composition and inverse is a member

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Matrix</th>
<th># DoF</th>
<th>Preserves</th>
<th>Icon</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation</td>
<td>$\begin{bmatrix} I &amp; t \end{bmatrix}$$_{2 \times 3}$</td>
<td>2</td>
<td>orientation</td>
<td></td>
</tr>
<tr>
<td>rigid (Euclidean)</td>
<td>$\begin{bmatrix} R &amp; t \end{bmatrix}$$_{2 \times 3}$</td>
<td>3</td>
<td>lengths</td>
<td></td>
</tr>
<tr>
<td>similarity</td>
<td>$\begin{bmatrix} sR &amp; t \end{bmatrix}$$_{2 \times 3}$</td>
<td>4</td>
<td>angles</td>
<td></td>
</tr>
<tr>
<td>affine</td>
<td>$\begin{bmatrix} A \end{bmatrix}$$_{2 \times 3}$</td>
<td>6</td>
<td>parallelism</td>
<td></td>
</tr>
<tr>
<td>projective</td>
<td>$\begin{bmatrix} \hat{H} \end{bmatrix}$$_{3 \times 3}$</td>
<td>8</td>
<td>straight lines</td>
<td></td>
</tr>
</tbody>
</table>

[source: R. Szeliski]
Projective Transformations

- Homography

\[
\begin{bmatrix}
    x' \\
    y' \\
    1
\end{bmatrix}
= \begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]

- Properties:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved
  - Closed under composition
  - Rectangle goes to quadrilateral
  - Affine transformation is a special case, where \( g = h = 0 \) and \( i = 1 \)

[Source: N. Snavely, slide credit: R. Urtasun]
What Transformation Happened To My DVD?

For planar objects:
- Viewpoint change for planar objects is a homography
- Affine transformation approximates viewpoint change for planar objects that are far away from camera
Homography

• Why should I care about homography?
• Now that I care, how should I estimate it?
• I want to understand the geometry behind homography. That is, why aren’t parallel lines mapped to parallel lines in oblique viewpoints? How did we get that equation for computing the homography?
Homography

- Why should I care about homography? Let’s answer this first
- Now that I care, how should I estimate it?
- I want to understand the geometry behind homography. That is, why aren’t parallel lines mapped to parallel lines in oblique viewpoints? How did we get that equation for computing the homography?
Homography

Why do we need homography? Can’t we just assume that the transformation is affine? The approximation on the right looks pretty decent to me...
Homography

• Why do we need homography? Can’t we just assume that the transformation is affine? The approximation on the right looks pretty decent to me...
• That’s right. If I want to detect (match) an object in a new viewpoint, an affine transformation is a relatively decent approximation
Homography

• Why do we need homography? Can’t we just assume that the transformation is affine? The approximation on the right looks pretty decent to me...

• That’s right. If I want to detect (match) an object in a new viewpoint, an affine transformation is a relatively decent approximation

• But for some applications I want to be more accurate. Which?
Application 1: a Little Bit of CSI

- Tom Cruise is taking an exam on Monday
Application 1: a Little Bit of CSI

- The professor keeps the exams in this office
Application 1: a Little Bit of CSI

- He enters (without permission) and takes a picture of the laptop screen
Application 1: a Little Bit of CSI

- His picture turns out to not be from a viewpoint he was shooting for (it’s difficult to take pictures while hanging)
- Can he still read the exam?
Warping an Image with a Global Transformation

Transformation $T$ is a coordinate-changing machine:

\[ [x', y'] = T(x, y) \]

$p = (x, y)$

$P' = (x', y')$

[Source: N. Snavely, slide credit: R. Urtasun]
Warping an Image with a Global Transformation

\[ p = (x, y) \]
\[ P' = (x', y') \]

Transformation \( T \) is a coordinate-changing machine:

\[ [x', y'] = T(x, y) \]

What does it mean that \( T \) is global?

- Is the same for any point \( p \)
- Can be described by just a few numbers (parameters)

[Source: N. Snavely, slide credit: R. Urtasun]
Warping an Image with a Global Transformation

• Example of warping for different transformations:

  what happens to a pixel \((x, y)\)?
Warping an Image with a Global Transformation

- Example of warping for different transformations:

  what happens to a pixel \((x, y)\)?
Warping an Image with a Global Transformation

- Example of warping for different transformations:

  - what happens to a pixel \((x, y)\)?
Warping an Image with a Global Transformation

- Example of warping for different transformations:

  - translation
  - rotation
  - aspect
  - affine

what happens to a pixel (x, y)?
Warping an Image with a Global Transformation

- Example of warping for different transformations:

  - what happens to a pixel \((x, y)\)?
Forward and Inverse Warping

• Forward Warping: Send each pixel \( f(x) \) to its corresponding location \((x', y') = T(x, y)\) in \( g(x', y') \)

```plaintext
procedure forwardWarp(f, h, out g):
    For every pixel \( x \) in \( f(x) \)
        1. Compute the destination location \( x' = h(x) \).
        2. Copy the pixel \( f(x) \) to \( g(x') \).
```
Forward and Inverse Warping

• Forward Warping: Send each pixel \( f(x) \) to its corresponding location \((x', y') = T(x, y)\) in \( g(x', y')\)

\[
\text{procedure forwardWarp}(f, h, \text{out } g):
\]

For every pixel \( x \) in \( f(x) \)

1. Compute the destination location \( x' = h(x) \).
2. Copy the pixel \( f(x) \) to \( g(x') \).

• May leave some holes in the target image.
Forward and Inverse Warping

Figure: Loren Arthur Schwarz
Forward and Inverse Warping

• Forward Warping: Send each pixel $f(x)$ to its corresponding location $(x', y') = T(x, y)$ in $g(x', y')$

```
procedure forwardWarp(f, h, out g):
    For every pixel $x$ in $f(x)$
        1. Compute the destination location $x' = h(x)$.
        2. Copy the pixel $f(x)$ to $g(x')$.
```

• May leave some holes in the target image.
• Inverse Warping: Each pixel at destination is sampled from original image

```
procedure inverseWarp(f, h, out g):
    For every pixel $x'$ in $g(x')$
        1. Compute the source location $x = \hat{h}(x')$
        2. Resample $f(x)$ at location $x$ and copy to $g(x')$
```
Forward and Inverse Warping

Figure: Loren Arthur Schwarz
Application 1: a Little Bit of CSI

- We want to transform the picture (plane) inside these 4 points into a rectangle (laptop screen)
Application 1: a Little Bit of CSI

Screen resolution is $900 \times 1440$

• We want it to look like this. How can we do this?
Application 1: a Little Bit of CSI

- A transformation that maps a projective plane (a quadrilateral) to another projective plane (another quadrilateral, in this case a rectangle) is a homography.
Application 1: a Little Bit of CSI

• If we compute the homography and warp the image according to it, we get this
Application 1: a Little Bit of CSI

- If we used affine transformation instead, we’d get this. Would be even worse if our picture was taken closer to the laptop.
Application 1: a Little More of CSI

What is the shape of the b/w floor pattern?

The floor (enlarged)

Slide from Antonio Criminisi
Application 1: a Little More of CSI

What is the shape of the b/w floor pattern?

Homography

The floor (enlarged)

Automatically rectified floor

Slide from Antonio Criminisi
Application 1: a Little More of CSI

From Martin Kemp *The Science of Art* (manual reconstruction)

Slide from Antonio Criminisi
Application 1: a Little More of CSI

*St. Lucy Altarpiece*, D. Veneziano

Slide from Criminisi
Application 1: a Little More of CSI

What is the (complicated) shape of the floor pattern?

*St. Lucy Altarpiece*, D. Veneziano
Slide from Criminisi

Automatically rectified floor
Application 1: a Little More of CSI

Automatic rectification

From Martin Kemp, The Science of Art (manual reconstruction)

Slide from Criminisi
A weird drawing

Holbein, “The Ambassadors”
A weird drawing

Holbein, “The Ambassadors”

What’s this???
A weird drawing

Holbein, “The Ambassadors”

rectified view

skull under anamorphic perspective
A weird drawing

Holbein, “The Ambassadors”

DIY: use a polished spoon to see the skull
Application 2: How Much do Soccer Players Run?
Application 2: How Much do Soccer Players Run?

- How many meters did this player run?
Application 2: How Much do Soccer Players Run?

• Field is planar. We know its dimensions (look on Wikipedia).
Application 2: How Much do Soccer Players Run?

• Let’s take the 4 corner points of the field
Application 2: How Much do Soccer Players Run?

- We need to compute a homography that maps them to these 4 corners.
Application 2: How Much do Soccer Players Run?

- We need to compute a homography that maps the 4 corners. Any other point from this plane (the field) also maps to the right with the same homography
Application 2: How Much do Soccer Players Run?

• Nice. What happened to the players?
Application 2: How Much do Soccer Players Run?

- We can now also transform the player’s trajectory → and we have it in meters!
Application 2: How Much do Soccer Players Run?

- If we used affine transformation... Our estimations of running would not be accurate!
Application 4: Panorama Stitching

Take a tripod, rotate camera and take pictures

[Source: Fernando Flores-Mangas]
Application 4: Panorama Stitching

[Source: Fernando Flores-Mangas]
Application 4: Panorama Stitching

• Each pair of images is related by homography! If we also moved the camera, this wouldn’t be true (next class)

[Source: Fernando Flores-Mangas]
We can use homographies when...

- The scene is planar

- The scene is relatively far off or has small (relative) depth variation (i.e., scene is approximately planar)
We can use homographies when...

• The scene is captured under camera rotation only (no translation or pose change)
Homography

- Why should I care about homography?
- Now that I care, how should I estimate it? Let’s do this now
- I want to understand the geometry behind homography. That is, why aren’t parallel lines mapped to parallel lines in oblique viewpoints? How did we get that equation for computing the homography?
Solving for Homographies

• Projective mapping between any two projection planes with the same centre of projection.
• Let \((x_i, y_i)\) be a point on the reference (model) image, and \((x'_i, y'_i)\) its match in the test image.
• A homography \(H\) maps \((x_i, y_i)\) to \((x'_i, y'_i)\):

\[
\begin{bmatrix}
ax'_i \\
ay'_i \\
a
\end{bmatrix} =
\begin{bmatrix}
h_{00} & h_{01} & h_{02} \\
h_{10} & h_{11} & h_{12} \\
h_{20} & h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i \\
1
\end{bmatrix}
\]
Solving for Homographies

• Projective mapping between any two projection planes with the same centre of projection
• Let \((x_i, y_i)\) be a point on the reference (model) image, and \((x'_i, y'_i)\) its match in the test image
• A homography \(H\) maps \((x_i, y_i)\) to \((x'_i, y'_i)\):

\[
\begin{bmatrix}
ax_i' \\
ay_i' \\
a
\end{bmatrix} =
\begin{bmatrix}
h_{00} & h_{01} & h_{02} \\
h_{10} & h_{11} & h_{12} \\
h_{20} & h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i \\
1
\end{bmatrix}
\]

• Expand matrix multiplication

\[
ax_i' = h_{00}x_i + h_{01}y_i + h_{02}
\]
\[
ay_i' = h_{10}x_i + h_{11}y_i + h_{12}
\]
\[
a = h_{20}x_i + h_{21}y_i + h_{22}
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Solving for Homographies

- Projective mapping between any two projection planes with the same centre of projection
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\begin{bmatrix}
x_i \\
y_i \\
1
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  ax'_i = h_{00}x_i + h_{01}y_i + h_{02}
  \]
  
  \[
  ay'_i = h_{10}x_i + h_{11}y_i + h_{12}
  \]
  
  \[
  a = h_{20}x_i + h_{21}y_i + h_{22}
  \]

- Divide out scale factor
  
  \[
  x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}
  \]
  
  \[
  y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}
  \]
Solving for Homographies

• Projective mapping between any two projection planes with the same centre of projection
• Let \((x_i, y_i)\) be a point on the reference (model) image, and \((x'_i, y'_i)\) its match in the test image
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\begin{bmatrix}
x_i \\
y_i \\
1
\end{bmatrix}
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• Expand matrix multiplication

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• Divide out scale factor

\[
x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}
\]
\[
y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}
\]

Can I rewrite this into a linear system in \(h\)?
Solving for Homographies

• From:

\[ x_i' = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}} \]
\[ y_i' = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}} \]

• We can easily get this:

\[ x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02} \]
\[ y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12} \]
Solving for Homographies

• From:

\[ x_i' = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}} \]

\[ y_i' = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}} \]

• We can easily get this:

\[ x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02} \]

\[ y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12} \]

• Rewriting it a little:

\[ h_{00}x_i + h_{01}y_i + h_{02} - x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = 0 \]

\[ h_{10}x_i + h_{11}y_i + h_{12} - y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = 0 \]
Solving for Homographies

• We can re-write these equations:

\[ h_{00}x_i + h_{01}y_i + h_{02} - x_i'(h_{20}x_i + h_{21}y_i + h_{22}) = 0 \]
\[ h_{10}x_i + h_{11}y_i + h_{12} - y_i'(h_{20}x_i + h_{21}y_i + h_{22}) = 0 \]

• As a linear system:

\[
\begin{bmatrix}
  x_i & y_i & 1 & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i' \\
  0 & 0 & 0 & x_i & y_i & 1 & -y_i'x_i & -y'y_i & -y_i'
\end{bmatrix} \begin{bmatrix}
  h_{00} \\
  h_{01} \\
  h_{02} \\
  h_{10} \\
  h_{11} \\
  h_{12} \\
  h_{20} \\
  h_{21} \\
  h_{22}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]
Exact case

- If we have exactly 4 matches, this gives 8 equations (how many columns?)

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  \vdots \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
  0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n'
\end{bmatrix}
\begin{bmatrix}
  h_{00} \\
  h_{01} \\
  h_{02} \\
  h_{10} \\
  h_{11} \\
  h_{12} \\
  h_{20} \\
  h_{21} \\
  h_{22}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]

\[A \quad 2n \times 9
\]

- Are the columns linearly dependent or independent?
Exact case

• If we have exactly 4 matches, this gives 8 equations (how many columns?)

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'y_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  & & \vdots & & & & & & \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'y_n & -x_n'y_n & -x_n' \\
  0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n'
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  h_{00} \\
  h_{01} \\
  h_{02} \\
  h_{10} \\
  h_{11} \\
  h_{12} \\
  h_{20} \\
  h_{21} \\
  h_{22}
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  2n
\end{bmatrix}
\]

• Are the columns linearly dependent or independent?
• Is there a null space?
Exact case

• If we have exactly 4 matches, this gives 8 equations (how many columns?)

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  \vdots \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
  0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n'
\end{bmatrix}
\begin{bmatrix}
h_{00} \\
h_{01} \\
h_{02} \\
h_{10} \\
h_{11} \\
h_{12} \\
h_{20} \\
h_{21} \\
h_{22}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
2n
\end{bmatrix}
\]

\[A_{2n \times 9}\]

• Are the columns linearly dependent or independent?
• Is there a null space?
• How does this relate to the homography?
Solving for Homographies

Taking all matches into account:

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  \vdots
\end{bmatrix}
\begin{bmatrix}
  \vdots \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
  0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n' \\
\end{bmatrix}
\begin{bmatrix}
  h_{00} \\
  h_{01} \\
  h_{02} \\
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  h_{11} \\
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\end{bmatrix}
= \begin{bmatrix}
  0 \\
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  2n
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\]
Solving for Homographies

- Taking all matches into account:

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  \vdots & & & & & & & & \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
  0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n'
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  h_{00} \\
  h_{01} \\
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\end{bmatrix} = \begin{bmatrix}
  0 \\
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  2n
\end{bmatrix}
\]

- Can be written as a least squares problem

\[
\min_h \|Ah\|_2^2
\]
Solving for Homographies

• Taking all matches into account:

\[
\begin{bmatrix}
 x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
 0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
 \vdots \\
 x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
 0 & 0 & 0 & x_n & y_n & 1 & -y_n'x_n & -y_n'y_n & -y_n'
\end{bmatrix}
\begin{bmatrix}
 h_{00} \\
 h_{01} \\
 h_{02} \\
 h_{10} \\
 h_{11} \\
 h_{12} \\
 h_{20} \\
 h_{21} \\
 h_{22}
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 2n
\end{bmatrix}
\]

\[A_{2nx9}\]

• Can be written as a least squares problem \(\min_{h} \|Ah\|_2^2\)

• Can we use the Moore-Penrose pseudo inverse here?
Solving for Homographies

• Taking all matches into account:

\[
\begin{bmatrix}
  x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1'x_1 & -x_1'y_1 & -x_1' \\
  0 & 0 & 0 & x_1 & y_1 & 1 & -y_1'x_1 & -y_1'y_1 & -y_1' \\
  \vdots \\
  x_n & y_n & 1 & 0 & 0 & 0 & -x_n'x_n & -x_n'y_n & -x_n' \\
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  h_{21} \\
  h_{22}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  2n
\end{bmatrix}
\]

• Can be written as a least squares problem

\[
\min_h \|Ah\|_2^2
\]

• But this is underdetermined, (8 degrees of freedom) how do we account for the unknown scale factor?
Solving for Homographies pt 2

• we solve the *homogeneous* least squares problem

\[
\min_h E = \|Ah\|_2^2 \\
\text{s.t. } \|h\|_2 = 1
\]

• Solve with eigenvalue decomposition of $A^T A$ or SVD
Reminder: Least Squares

Convert the system to a linear least-squares problem:

\[ E_{\text{LLS}} = \|Ax - b\|^2 \]

Expand the error:

\[ E_{\text{LLS}} = x^\top (A^\top A)x - 2x^\top (A^\top b) + \|b\|^2 \]

Minimize the error:

Set derivative to 0 \( (A^\top A)x = A^\top b \)

Solve for \( x \)

\[ x = (A^\top A)^{-1}A^\top b \]

In Matlab:

\[ x = A \backslash b \]

Note: You almost never want to compute the inverse of a matrix.
Singular Value Decomposition

\[ A = U \Sigma V^T \]

\[ = \sum_{i=1}^{9} \sigma_i u_i v_i^T \]
General form of total least squares

(WARNING: change of notation. $\mathbf{x}$ is a vector of parameters!)

$$E_{\text{TLS}} = \sum_i (a_i \mathbf{x})^2$$

$$= \| \mathbf{Ax} \|^2 \quad \text{(matrix form)}$$

$$\| \mathbf{x} \|^2 = 1 \quad \text{constraint}$$

minimize $\| \mathbf{Ax} \|^2$

subject to $\| \mathbf{x} \|^2 = 1$

Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

(\text{equivalent})

Solution is the column of $\mathbf{V}$ corresponding to smallest singular value

$$\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^\top$$
Solving for $H$ using DLT

(Direct linear transformation: algorithm for solving homographies)

Given $\{x_i, x'_i\}$ solve for $H$ such that $x' = Hx$

1. For each correspondence, create $2 \times 9$ matrix $A_i$

2. Concatenate into single $2n \times 9$ matrix $A$

3. Compute SVD of $A = U \Sigma V^\top$

4. Store singular vector of the smallest singular value $h = v_i$

5. Reshape to get $H$
Image Alignment Algorithm: Homography

• Given images I and J
  • How can we find the alignment between images?

1. Compute image features for I and J
2. Match features between I and J
3. Compute homography transformation A between I and J
Image Alignment Algorithm: Homography

• Given images I and J
  • How can we find the alignment between images?
Panorama Stitching: Example 1

- Compute the matches

[Source: R. Queiroz Feitosa]
Panorama Stitching: Example 1

- Estimate the homography and warp

[Source: R. Queiroz Feitosa]
Panorama Stitching: Example 1

• Stitch

[Source: R. Queiroz Feitosa]
Panorama Stitching: Example 2

[Source: Fernando Flores-Mangas]
Panorama Stitching: Example 2

[Source: Fernando Flores-Mangas]
Panorama Stitching: Example 2

Laplacian Pyramid Blending \( \downarrow \) seams not visible anymore

[Brown & Lowe; ICCV 2003] google "Lowe Brown Autostitch"

[Source: Fernando Flores-Mangas]
The image correspondence pipeline

1. Feature point detection
   - Detect blobs using, e.g., multi-scale LoG

2. Feature point description
   - Describe features using the SIFT descriptor.

3. Feature matching
How do we estimate homographies when we have both good and bad correspondences?
Random Sample Consensus (RANSAC)
**Algorithm:**

1. Sample (randomly) the number of points required to fit the model
2. Solve for model parameters using samples
3. Score by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence
Algorithm:

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Repeat 1-3 until the best model is found with high confidence
How to choose parameters?

• Number of samples N
  – Choose N so that, with probability p, at least one random sample is free from outliers
    (e.g. p=0.99) (outlier ratio: e )
How to choose parameters?

• Number of samples $N$
  – Choose $N$ so that, with probability $p$, at least one random sample is free from outliers (e.g. $p=0.99$) (outlier ratio: $e$)

• Number of sampled points $s$
  – Minimum number needed to fit the model
How to choose parameters?

• Number of samples \( N \)
  
  – Choose \( N \) so that, with probability \( p \), at least one random sample is free from outliers (e.g. \( p=0.99 \)) (outlier ratio: \( e \))

• Number of sampled points \( s \)
  
  – Minimum number needed to fit the model

• Distance threshold \( \delta \)
  
  – Choose \( \delta \) so that a good point with noise is likely (e.g., prob=0.95) within threshold
  
  – Zero-mean Gaussian noise with std. dev. \( \sigma \): \( t^2=3.84\sigma^2 \)
How to choose parameters?

• Number of samples $N$
  – Choose $N$ so that, with probability $p$, at least one random sample is free from outliers (e.g. $p=0.99$) (outlier ratio: $e$)

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• Distance threshold $\delta$
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  – Zero-mean Gaussian noise with std. dev. $\sigma$: $t^2=3.84\sigma^2$

$$N = \frac{\log(1 - p)}{\log \left(1 - (1 - e)^s\right)}$$

- $p =$ desired probability that we get a good sample
- $s =$ number of points in a sample
- $N =$ number of samples (we want to compute this)
- $e =$ probability that a point is an outlier
How to choose parameters?

• Number of samples \( N \)
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\[
N = \frac{\log(1 - p)}{\log \left(1 - (1 - e)^s\right)}
\]

\( p = \) desired probability that we get a good sample
\( s = \) number of points in a sample
\( N = \) number of samples (we want to compute this)
\( e = \) probability that a point is an outlier

can you derive this? Hint: start with writing the probability that you choose one inlier point, then that you choose ‘s’ inlier points...
Given two images...

find matching features (e.g., SIFT) and a translation transform
Matched points will usually contain bad correspondences

how should we estimate the transform?
LLS will find the ‘average’ transform

solution is corrupted by bad correspondences
Use RANSAC

How many correspondences to compute translation transform?
Need only one correspondence, to find translation model
Pick one correspondence, count inliers
Pick one correspondence, count inliers

2 inliers
Pick one correspondence, count inliers
Pick one correspondence, count inliers

5 inliers
Pick one correspondence, count inliers

Pick the model with the highest number of inliers!
Estimating homography using RANSAC

• RANSAC loop
  1. Get point correspondences (randomly)
Estimating homography using RANSAC

- RANSAC loop
  1. Get four point correspondences (randomly)
  2. Compute H using DLT
Estimating homography using RANSAC

• RANSAC loop
  1. Get four point correspondences (randomly)
  2. Compute H using DLT
  3. Count inliers
     - Keep if largest number of inliers
     - Recompute H using all inliers
Estimating homography using RANSAC

- RANSAC loop
  1. Get four point correspondences (randomly)
  2. Compute H using DLT
  3. Count inliers
  4. Keep H if
Estimating homography using RANSAC

• RANSAC loop
  1. Get four point correspondences (randomly)
  2. Compute H using DLT
  3. Count inliers
  4. Keep H if largest number of inliers
• Recompute H using all inliers
The image correspondence pipeline

1. Feature point detection
   • Detect corners using the Harris corner detector.

2. Feature point description
   • Describe features using the Multi-scale oriented patch descriptor.

3. Feature matching and homography estimation
   • Do both simultaneously using RANSAC.
Summary – Stuff You Need To Know

- A homography is a mapping between projective planes
- You need at least 4 correspondences (matches) to compute it
- How do we assemble and solve the constrained least squares system?
- What is the RANSAC algorithm and how can I apply it to a model fitting problem?

OpenCV (Python):
- Affine transformation/warp:
  - getAffineTransform(pts_src, pts_dst)
  - warpAffine
- Perspective transformation/warp:
  - either: getPerspectiveTransform (pts_src, pts_dst) (without RANSAC)
  - or: h, status = cv2.findHomography(pts_src, pts_dst) (with RANSAC)
  - im_dst = warpPerspective(im_src, h, size)
## Birdseye View on What We Learned So Far

<table>
<thead>
<tr>
<th>Problem</th>
<th>Detection</th>
<th>Description</th>
<th>Matching</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find Planar Distinctive Objects</td>
<td>Scale Invariant Interest Points</td>
<td>Local feature: SIFT</td>
<td>All features to all features + Affine / Homography</td>
</tr>
</tbody>
</table>
More than one image

• We’re in 2023...

• Think not (only) what you can do with one image, but what lots and lots of images can do for you
More than one image

• We’re in 2023...

• Think not (only) what you can do with one image, but what lots and lots of images can do for you

• Would our current matching method work with lots of data?
Looking forward (object detection)

• So far we matched a known object in a new viewpoint
• What if we have to match an object to LOTS of images? Or LOTS of objects to one image?
• We’ll discuss this in a few weeks (object recognition)
Next Time: Camera Models