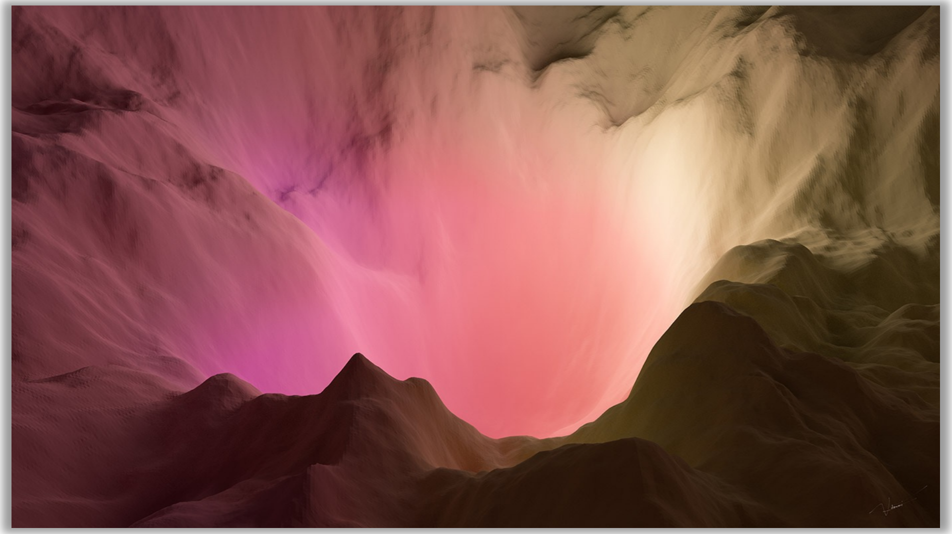
Solving Regularized Inverse Problems with ADMM



losslandscape.com

CSC2529

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cs.toronto.edu/~lindell/teaching/2529

*slides adapted from Gordon Wetzstein

Announcements

- HW5 due Wednesday 1/11
- HW6 is out (last one) & problem session tomorrow

- No class next week (reading week)
- Proposal due in 2.5 weeks!

- See website for all office hours/problem session dates

Overview

- Brief review of deconvolution with inverse/Wiener filtering
- A Bayesian perspective of inverse problems
- Image priors/regularization and total variation
- The ADMM method
- Image deconvolution with ADMM
- Compressive imaging

Must read: course notes on Deconvolution and Compressive Imaging!

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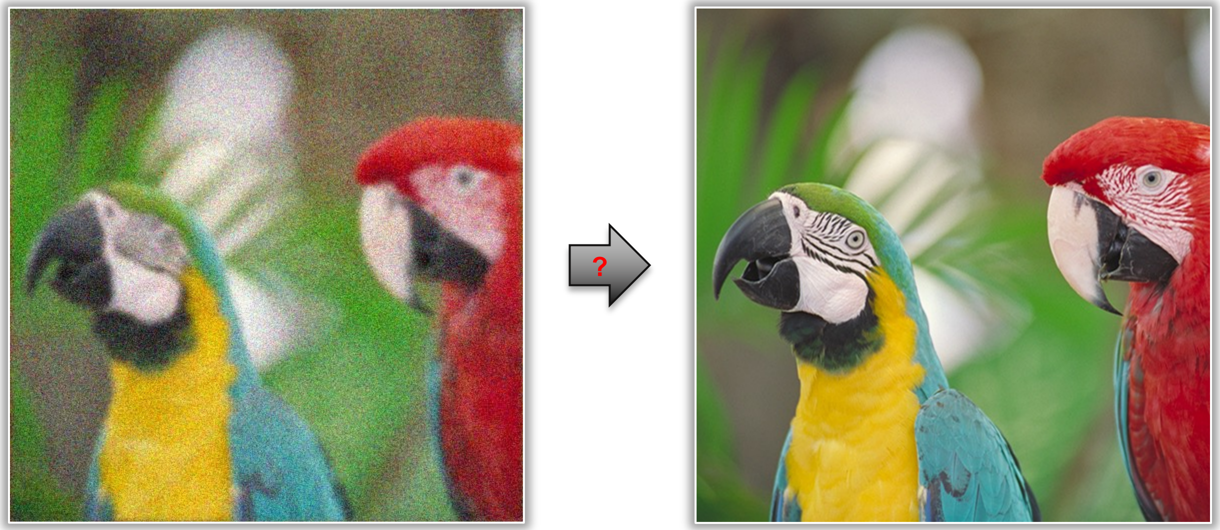
Image Deconvolution – Brief Review



Given: blurry & noisy image

Desired: sharp & noise-free image

Image Deconvolution – Brief Review

- Image formation model: $b = c * x + \eta$

2D measurements

additive noise

known 2D convolution kernel

2D target image

Image Deconvolution – Brief Review

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- Convolution theorem:

Image Deconvolution – Brief Review

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- Inverse filtering: $\tilde{x}_{\mathrm{if}} = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{b\}}{\mathcal{F}\{c\}}\right\}$
- Wiener filtering: $\tilde{x}_{\mathrm{wf}} = \mathcal{F}^{-1}\left\{\frac{|\mathcal{F}\{c\}|^2}{|\mathcal{F}\{c\}|^2 + {}^{1}/_{SNR}} \cdot \frac{\mathcal{F}\{b\}}{\mathcal{F}\{c\}}\right\}$

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- Duality of “signal processing” and “algebraic” interpretation:

$$b = c * x \;\Leftrightarrow\; \boldsymbol{b} = \boldsymbol{C}\boldsymbol{x} \qquad \boldsymbol{C} \in \mathbb{R}^{N \times N}, \quad \boldsymbol{b}, \boldsymbol{x} \in \mathbb{R}^{N}$$

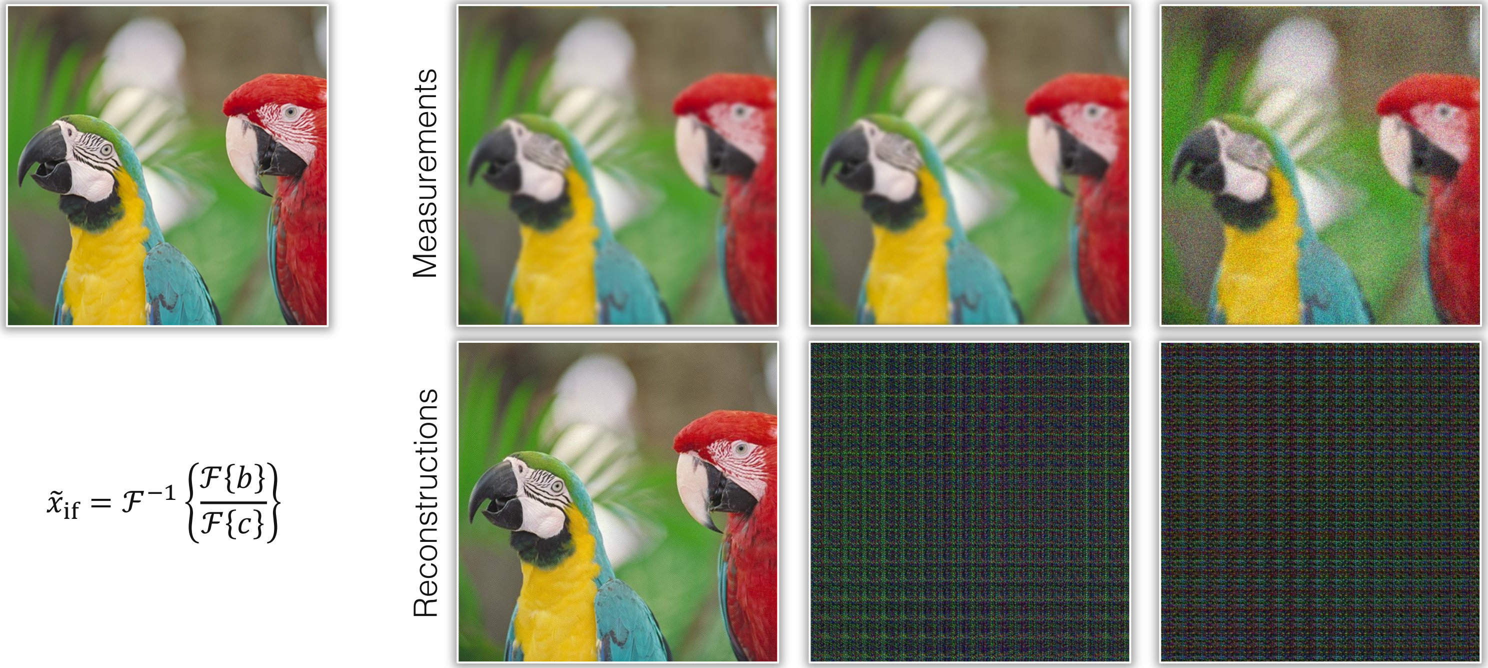
Image Deconvolution – Inverse Filtering

Ground Truth

No Noise

σ=0.1

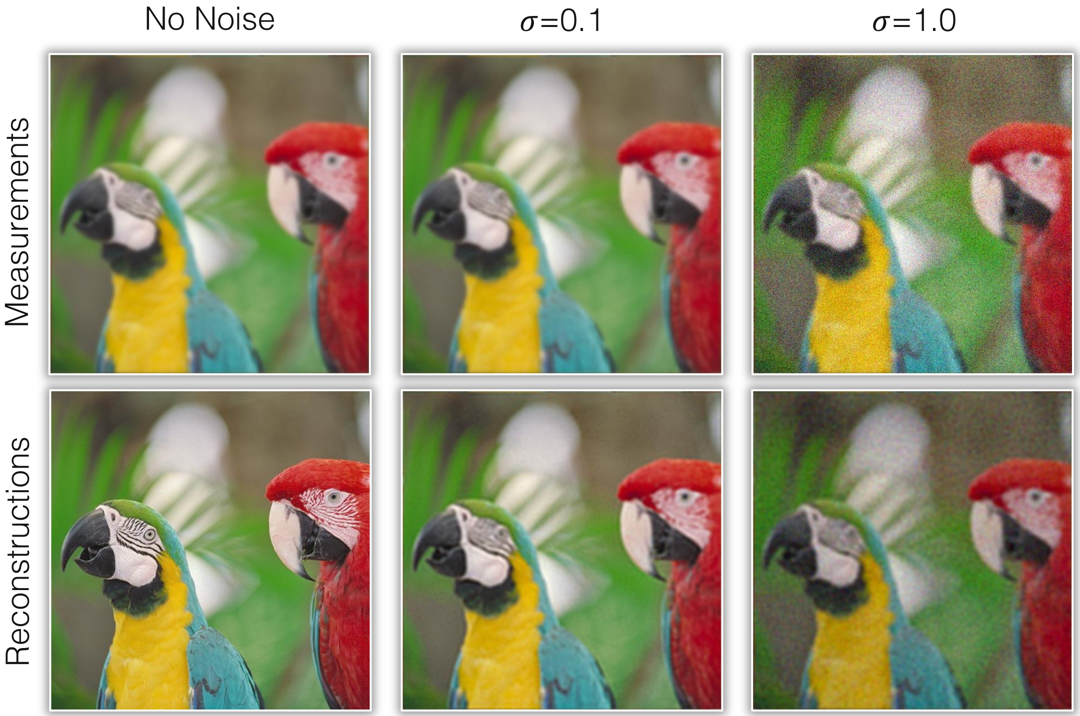
σ=1.0



$$\tilde{x}_{\mathrm{if}} = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{b\}}{\mathcal{F}\{c\}}\right\}$$

Image Deconvolution – Wiener Filtering

Ground Truth



$$\tilde{x}_{\mathrm{wf}} = \mathcal{F}^{-1}\left\{ \frac{|\mathcal{F}\{c\}|^2}{|\mathcal{F}\{c\}|^2 + {}^{1}/_{SNR}} \cdot \frac{\mathcal{F}\{b\}}{\mathcal{F}\{c\}} \right\}$$

Image Deconvolution

- Problem: this is an ill-posed inverse problem, i.e., there are infinitely many solutions that satisfy the measurements

- Need some way to determine how “desirable” any one of these feasible solutions is → need an **image prior**

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A Bayesian Perspective of Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^{M}, \boldsymbol{x} \in \mathbb{R}^{N}, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

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- Interpret as random variables:

$$\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{x}_i, 0),\ \boldsymbol{\eta}_i \sim \mathcal{N}(0, \sigma^2)$$

$$\boldsymbol{b}_i \sim \mathcal{N}\left((\boldsymbol{A}\boldsymbol{x})_i, \sigma^2\right)$$

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- Probability of observation i:
$$p(\boldsymbol{b}_i | \boldsymbol{x}_i, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\boldsymbol{b}_i - (\boldsymbol{A}\boldsymbol{x})_i)^2}{2\sigma^2}}$$

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- Joint probability of all observations:

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- Joint probability of all observations:
$$p(\boldsymbol{b} | \boldsymbol{x}, \sigma) = \prod_{i=1}^{M} p(\boldsymbol{b}_i | \boldsymbol{x}_i, \sigma) \propto e^{-\frac{\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2}{2\sigma^2}}$$

A Bayesian Perspective of Inverse Problems

- Bayes' rule: $$p(\boldsymbol{x}|\boldsymbol{b},\sigma) = \frac{p(\boldsymbol{b}|\boldsymbol{x},\sigma)\boldsymbol{p}(\boldsymbol{x})}{p(\boldsymbol{b})} \propto p(\boldsymbol{b}|\boldsymbol{x},\sigma)p(\boldsymbol{x})$$

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posterior

image formation model

prior

A Bayesian Perspective of Inverse Problems

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- Maximum-a-posterior (MAP) solution:

$$\boldsymbol{x}_{MAP} = \arg\min_x \; -\log\big(p(\boldsymbol{x}|\boldsymbol{b},\sigma)\big)$$

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posterior — image formation model — prior

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$$= \arg\min_x \; \frac{1}{2\sigma^2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \Psi(\boldsymbol{x})$$

A Bayesian Perspective of Inverse Problems

- Terminology: $$\underbrace{\Psi(\boldsymbol{x})}_{\text{regularizer}} = -\log\big(\underbrace{p(\boldsymbol{x})}_{\text{prior}}\big)$$

$$\boldsymbol{x}_{MAP} = \arg\min_{\boldsymbol{x}} \underbrace{\frac{1}{2\sigma^2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2}_{\text{data fidelity term}} + \underbrace{\Psi(\boldsymbol{x})}_{\text{regularization term}}$$

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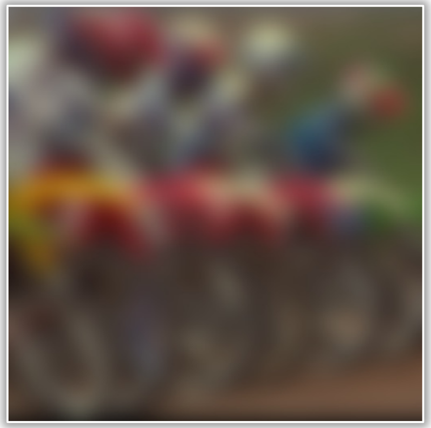
Examples of Image Priors / Regularizers

blurry stuff



Promote smoothness!

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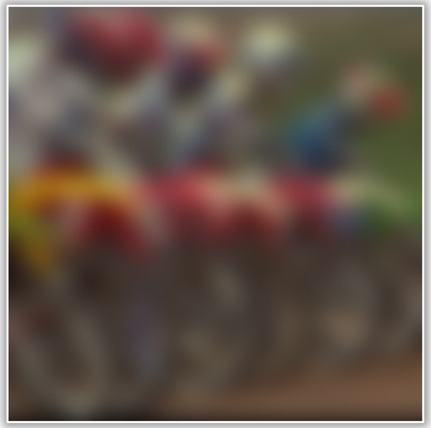

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$$\Psi(\boldsymbol{x}) = ||\Delta \boldsymbol{x}||_2$$

Laplace operator

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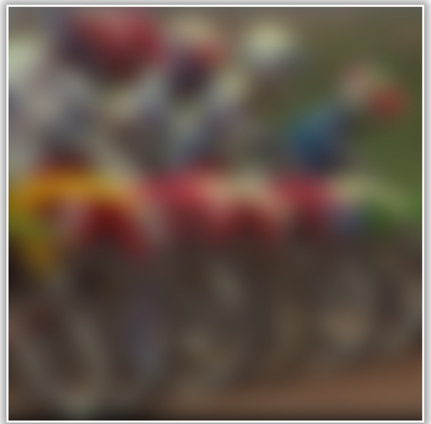

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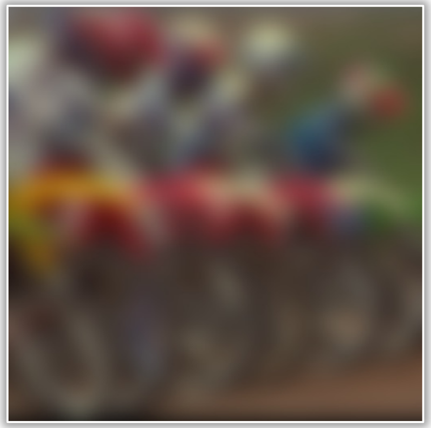

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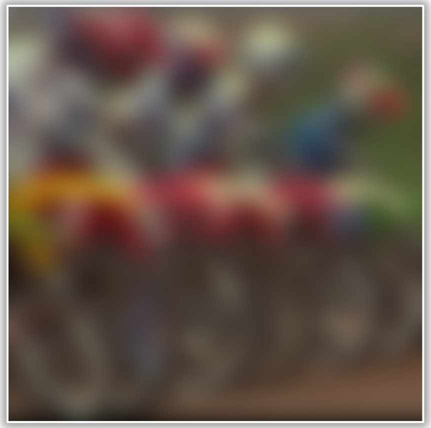

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"natural" image

Promote sparse gradients!

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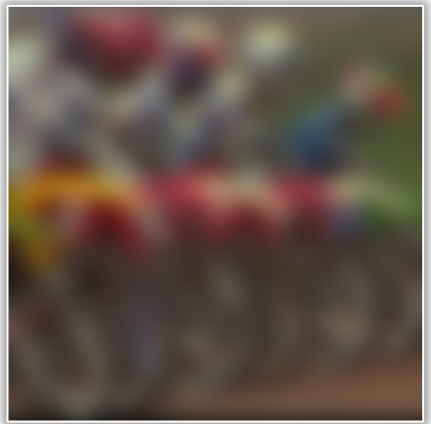

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"natural" image

Promote sparse gradients!

$$\Psi(\boldsymbol{x}) = \mathrm{TV}(\boldsymbol{x})$$

Total Variation (TV)

express (forward finite difference)
gradient as convolution

$\boldsymbol{x}$

$$\boldsymbol{D}_x\boldsymbol{x} = d_x * x, d_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

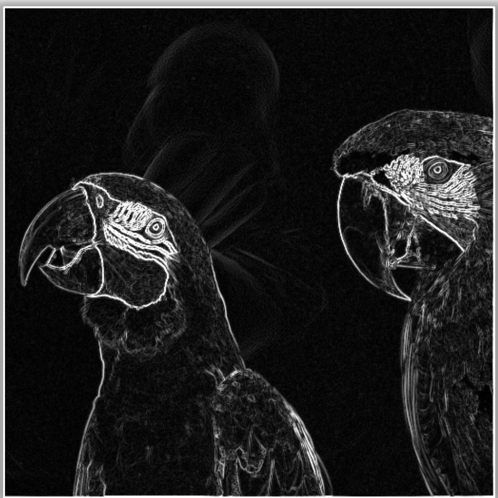
$$\boldsymbol{D}_y\boldsymbol{x} = d_y * x, d_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Total Variation (TV)

$\boldsymbol{x}$

better: isotropic

$$\sqrt{(\boldsymbol{D}_x\boldsymbol{x})_i^2 + (\boldsymbol{D}_y\boldsymbol{x})_i^2}$$



easier: anisotropic

$$\sqrt{(\boldsymbol{D}_x\boldsymbol{x})_i^2} + \sqrt{(\boldsymbol{D}_y\boldsymbol{x})_i^2}$$

Total Variation (TV)

- Examples are mostly black, indicating that gradient magnitudes are close to 0 → natural images have sparse gradients!

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$$\mathrm{TV}_{\mathrm{anisotropic}}(\boldsymbol{x}) = \|\boldsymbol{D}_x\boldsymbol{x}\|_1 + \left\|\boldsymbol{D}_y\boldsymbol{x}\right\|_1 = \sum_{i=1}^{N} |(\boldsymbol{D}_x\boldsymbol{x})_i| + \left|\left(\boldsymbol{D}_y\boldsymbol{x}\right)_i\right| = \sum_{i=1}^{N} \sqrt{(\boldsymbol{D}_x\boldsymbol{x})_i^2} + \sqrt{\left(\boldsymbol{D}_y\boldsymbol{x}\right)_i^2}$$

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- This intuition is well-captured by the TV pseudo-norm, either anisotropic or isotropic:

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$$\mathrm{TV}_{\mathrm{isotropic}}(\boldsymbol{x}) = \|\boldsymbol{D}\boldsymbol{x}\|_{2,1} = \sum_{i=1}^{N} \left\| \begin{bmatrix} (\boldsymbol{D}_x\boldsymbol{x})_i \\ \left(\boldsymbol{D}_y\boldsymbol{x}\right)_i \end{bmatrix} \right\|_2 = \sum_{i=1}^{N} \sqrt{(\boldsymbol{D}_x\boldsymbol{x})_i^2 + \left(\boldsymbol{D}_y\boldsymbol{x}\right)_i^2}$$

Total Variation (TV)

The TV pseudo-norm is one of the most popular regularization schemes for natural images!

Extensions to make it more general or applicable for other data:

- Hyper-Laplacian: Levin et al. 2009, Krishnan & Fergus 2009
- Total generalized variation: Bredies et al. 2009
- Frobenius norm of Hessian: Lefkimmiatis et al. 2003
- …

How to solve inverse problem that use these regularizers?

Solving Regularized Inverse Problem

- Objective or "loss" function of general inverse problem:

$$\text{minimize}_x \ \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\Psi(\boldsymbol{x})$$

weight of regularizer

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- Practical #1 go-to solution: Adam solver implemented in PyTorch

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- 3 simple steps, will explore in problem session & homework:
  1. Implement evaluation of loss function
  2. Set hyperparameters, including learning rate
  3. Run

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- 3 simple steps, will explore in problem session & homework:
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  3. Run
- The "fine print": convenient but doesn't always converge well

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Regularized Image Reconstruction

$$\underset{\{x\}}{\text{minimize}} \ \frac{1}{2}\left\|b - Ax\right\|_2^2 + \lambda\Gamma(x)$$

data fidelity term

some image prior, such as ℓ_1 norm or others

Regularized Image Reconstruction

$$\underset{\{x\}}{\text{minimize}} \quad \frac{1}{2}\|b - Ax\|_2^2 + \lambda\Gamma(x)$$

$$\underset{\{x\}}{\text{minimize}} \quad \underbrace{\frac{1}{2}\|b - Ax\|_2^2}_{f(x)} + \underbrace{\lambda\Gamma(z)}_{g(z)}$$

$$\text{subject to} \quad Kx - z = 0$$

- split into two parts → mathematically equivalent

Regularized Image Reconstruction

$$\underset{\{x\}}{\text{minimize}} \quad \underbrace{\frac{1}{2}\left\|b - Ax\right\|_2^2}_{f(x)} + \underbrace{\lambda\Gamma(z)}_{g(z)}$$

$$\text{subject to} \quad Kx - z = 0$$

$$L(x, z, y) = f(x) + g(z) + y^T(Kx - z)$$

- Lagrangian

Regularized Image Reconstruction

- Lagrangian

$$L_\rho(x,z,y) = f(x) + g(z) + y^T(Kx - z)$$

Optimal if all partial derivatives are zero!

$$\nabla_{x,z,y} L = 0$$

Regularized Image Reconstruction

- Lagrangian

$$L_\rho(x,z,y) = f(x) + g(z) + y^T(Kx - z)$$

Also implies:

$$\nabla_{x,z,y} L = 0 \Leftrightarrow \nabla_{x,z} f(x) + g(z) = y^T \nabla_{x,z}(Kx - z)$$

$$Kx - z = 0$$

Regularized Image Reconstruction

- Lagrangian

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Regularized Image Reconstruction

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Regularized Image Reconstruction

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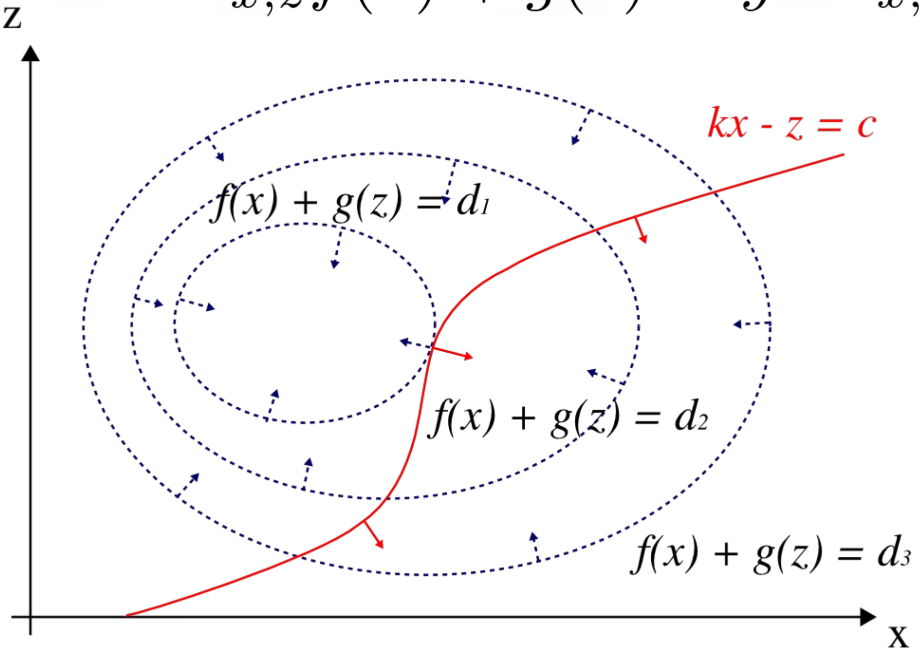

Regularized Image Reconstruction

- Augmented Lagrangian

$$L_\rho(x,z,y) = f(x) + g(z) + y^T(Kx - z) + \frac{\rho}{2}\|Kx - z\|_2^2$$

Regularized Image Reconstruction

- Augmented Lagrangian

$$L_\rho(x,z,y) = f(x) + g(z) + y^T(Kx - z) + \boxed{\frac{\rho}{2}\|Kx - z\|_2^2}$$

We add quadratic penalty: improves convergence properties compared to standard Lagrangian

ρ is called the “penalty parameter” (see Boyd 2011)

Regularized Image Reconstruction

- Scaled dual form of the augmented Lagrangian
  - Given by some algebraic manipulation (easier form to work with)

$$L_\rho(x, z, y) = f(x) + g(z) + \frac{\rho}{2}\|Kx - z + u\|^2 + \frac{\rho}{2}\|u\|^2$$

Where $u = y/\rho$ is the scaled dual variable

Regularized Image Reconstruction

- Recap
  - Split the objective function and enforce consistency with constraint
  - handle constraints using Lagrangian
  - Lagrangian -> Augmented Lagrangian -> scaled dual form

$$L_\rho(x, z, y) = f(x) + g(z) + \frac{\rho}{2}\|Kx - z + u\|^2 + \frac{\rho}{2}\|u\|^2$$

Regularized Image Reconstruction

- Recap
  - Split the objective function and enforce consistency with constraint
  - handle constraints using Lagrangian
  - Lagrangian -> Augmented Lagrangian -> scaled dual form
  - optimize using ADMM!

$$L_\rho(x, z, y) = f(x) + g(z) + \frac{\rho}{2}\|Kx - z + u\|^2 + \frac{\rho}{2}\|u\|^2$$

Regularized Image Reconstruction

$$L_\rho(x,z,y) = f(x) + g(z) + \frac{\rho}{2}\|Kx - z + u\|^2 + \frac{\rho}{2}\|u\|^2$$

repeat until converged

$$x \leftarrow \quad \mathrm{prox}_{\|\cdot\|_2,\rho}(v) = \underset{\{x\}}{\arg\min}\, L_\rho(x,z,y)$$

$$z \leftarrow \quad \mathrm{prox}_{\Gamma,\rho}(v) = \underset{\{z\}}{\arg\min}\, L_\rho(x,z,y)$$

$\{z\}$ z

$$u \leftarrow \quad u + Kx - z$$

- iterative updates - ADMM

Regularized Image Reconstruction

$$\mathbf{prox}_{\lambda f}(v) = \underset{x}{\operatorname{argmin}} \left(f(x) + (1/2\lambda)\|x - v\|_2^2 \right)$$

gives proximal point of v with respect to f

- finds a value of that is close to v and minimum of f, or moves to the domain of f

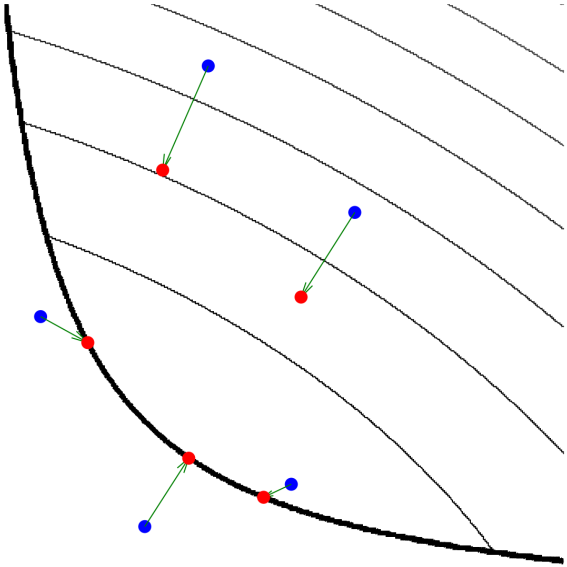

Figure 1.1: Evaluating a proximal operator at various points.

Regularized Image Reconstruction

repeat until converged

$$x \leftarrow \quad \mathrm{prox}_{\|\cdot\|_2,\rho}(v) = \underset{\{x\}}{\arg\min}\, L_\rho(x,z,y) = \underset{\{x\}}{\arg\min}\, \frac{1}{2}\|Ax-b\|_2^2 + \frac{\rho}{2}\|Kx-v\|, \; v = z-u$$

$$z \leftarrow \quad \mathrm{prox}_{\Gamma,\rho}(v) = \underset{\{z\}}{\arg\min}\, L_\rho(x,z,y) = \underset{\{z\}}{\arg\min}\, \lambda\Gamma(z) + \frac{\rho}{2}\|v-z\|, \; v = Kx+u$$

$$u \leftarrow \quad u + Kx - z$$

- iterative updates - ADMM

Regularized Image Reconstruction

$$\operatorname{prox}_{\|\cdot\|_2,\rho}(v) = \underset{\{x\}}{\arg\min}\ \frac{1}{2}\|Ax - b\|_2^2 + \frac{\rho}{2}\|Kx - v\|$$

… see notes ...

Regularized Image Reconstruction

$$\mathrm{prox}_{\|\cdot\|_2,\rho}(v) = \underset{\{x\}}{\arg\min}\, \frac{1}{2}\|Ax-b\|_2^2 + \frac{\rho}{2}\|Kx-v\|$$

… see notes ...

$$\mathrm{prox}_{\|\cdot\|_2,\rho}(v) = \Bigg(\underbrace{A^T A + \rho K^T K}_{\tilde{A}}\Bigg)^{-1}\Bigg(\underbrace{A^T b + \rho K^T v}_{\tilde{b}}\Bigg)$$

- x-update: solve $\tilde{A}x = \tilde{b}$
- symmetric, positive definite matrix → conjugate gradient method

Overview

- Brief review of deconvolution with inverse/Wiener filtering
- A Bayesian perspective of inverse problems
- Image priors/regularization and total variation
- The Alternating Direction Method of Multipliers (ADMM)
- Image deconvolution with ADMM
- Compressive imaging

ADMM for Image Deconvolution with TV

Generic: $$L_\rho(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{Kx} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

$\boldsymbol{x} \in \mathbb{R}^N$	unknown sharp image
$\boldsymbol{C} \in \mathbb{R}^{N \times N}$	circulant convolution matrix for known kernel $\boldsymbol{c}$
$\boldsymbol{z}, \boldsymbol{u} \in \mathbb{R}^{2N}$	slack/dual variable, twice the size of $\boldsymbol{x}$!
$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_x \\ \boldsymbol{D}_y \end{bmatrix} \in \mathbb{R}^{2N \times N}$	finite difference gradients, horizontal & vertical

ADMM for Image Deconvolution with TV

Generic: $$L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

Deconv: $$L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

$\boldsymbol{x} \in \mathbb{R}^N$ unknown sharp image

$\boldsymbol{C} \in \mathbb{R}^{N\times N}$ circulant convolution matrix for known kernel $\boldsymbol{c}$

$\boldsymbol{z},\boldsymbol{u} \in \mathbb{R}^{2N}$ slack/dual variable, twice the size of $\boldsymbol{x}$!

$\boldsymbol{D} = \begin{bmatrix}\boldsymbol{D}_x \\ \boldsymbol{D}_y\end{bmatrix} \in \mathbb{R}^{2N\times N}$ finite difference gradients, horizontal & vertical

ADMM for Image Deconvolution with TV

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{Cx} - \boldsymbol{b}\|_2^2 + \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{Dx} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

```
while not converged:
```

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_x \frac{1}{2}\|\boldsymbol{Cx} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{Dx} - \boldsymbol{v}\|_2^2, \boldsymbol{v} = \boldsymbol{z} - \boldsymbol{u}$$

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_z \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{z}\|_2^2, \boldsymbol{v} = \boldsymbol{Dx} + \boldsymbol{u}$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{v}\|_2^2$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v}\|_2^2$$

reformulate

$$= \frac{1}{2}(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b})^T(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}) + \frac{\rho}{2}(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})^T(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v}\|_2^2$$

reformulate

$$= \frac{1}{2}(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b})^T(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}) + \frac{\rho}{2}(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})^T(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})$$

$$= \frac{1}{2}\left(\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{b} + \boldsymbol{b}^T\boldsymbol{b}\right) + \frac{\rho}{2}\left(\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{v} + \boldsymbol{v}^T\boldsymbol{v}\right)$$

ADMM for Image Deconvolution with TV

<u>$\boldsymbol{x}$ – update:</u>

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v}\|_2^2$$

reformulate

$$= \frac{1}{2}(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b})^T(\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}) \ + \frac{\rho}{2}(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})^T(\boldsymbol{D}\boldsymbol{x}-\boldsymbol{v})$$

$$= \frac{1}{2}\left(\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{b} + \boldsymbol{b}^T\boldsymbol{b}\right) \ + \frac{\rho}{2}\left(\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{v} + \boldsymbol{v}^T\boldsymbol{v}\right)$$

find solution by setting gradient to 0

$$0 = \nabla_x f(\boldsymbol{x}) = \boldsymbol{C}^T\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}^T\boldsymbol{b} \ + \rho\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{x} \ - \ \rho\boldsymbol{D}^T\boldsymbol{v}$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{Cx}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{Dx}-\boldsymbol{v}\|_2^2$$

reformulate

$$= \frac{1}{2}(\boldsymbol{Cx}-\boldsymbol{b})^T(\boldsymbol{Cx}-\boldsymbol{b}) + \frac{\rho}{2}(\boldsymbol{Dx}-\boldsymbol{v})^T(\boldsymbol{Dx}-\boldsymbol{v})$$

$$= \frac{1}{2}\left(\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{Cx} - 2\boldsymbol{x}^T\boldsymbol{C}^T\boldsymbol{b} + \boldsymbol{b}^T\boldsymbol{b}\right) + \frac{\rho}{2}\left(\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{Dx} - 2\boldsymbol{x}^T\boldsymbol{D}^T\boldsymbol{v} + \boldsymbol{v}^T\boldsymbol{v}\right)$$

find solution by setting gradient to 0

$$0 = \nabla_x f(\boldsymbol{x}) = \boldsymbol{C}^T\boldsymbol{Cx} - \boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{Dx} - \rho\boldsymbol{D}^T\boldsymbol{v}$$

closed-form solution

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \operatorname{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{v}\|_2^2$$

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{v}\|_2^2$$

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)$$

Exploit duality of algebraic & signal processing interpretation

$$\boldsymbol{C}^T\boldsymbol{C} \Leftrightarrow \mathcal{F}^{-1}\{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\}\}$$

$$\boldsymbol{D}^T\boldsymbol{D} \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{d_x\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{d_y\}\right\}$$

$$\boldsymbol{D}^T\boldsymbol{z} = \boldsymbol{D}_x^T\boldsymbol{v}_1 + \boldsymbol{D}_y^T\boldsymbol{v}_2 \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{d_x\} *\cdot \mathcal{F}\{v_1\} + \mathcal{F}\{d_y\} *\cdot \mathcal{F}\{v_2\}\right\}$$

$$\boldsymbol{C}^T\boldsymbol{b} \Leftrightarrow \mathcal{F}^{-1}\{\mathcal{F}\{c\} *\cdot \mathcal{F}\{b\}\}$$

ADMM for Image Deconvolution with TV

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{Cx}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{Dx}-\boldsymbol{v}\|_2^2$$

$$\boldsymbol{x} \leftarrow \underline{\left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)}^{-1}\underline{\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)}$$

Exploit duality of algebraic & signal processing interpretation

$$\boldsymbol{C}^T\boldsymbol{C} \Leftrightarrow \mathcal{F}^{-1}\{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\}\}$$

$$\boldsymbol{D}^T\boldsymbol{D} \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{d_x\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{d_y\}\right\}$$

$$\boldsymbol{D}^T\boldsymbol{z} = \boldsymbol{D}_x^T\boldsymbol{v}_1 + \boldsymbol{D}_y^T\boldsymbol{v}_2 \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{d_x\} *\cdot \mathcal{F}\{v_1\} + \mathcal{F}\{d_y\} *\cdot \mathcal{F}\{v_2\}\right\}$$

$$\boldsymbol{C}^T\boldsymbol{b} \Leftrightarrow \mathcal{F}^{-1}\{\mathcal{F}\{c\} *\cdot \mathcal{F}\{b\}\}$$

$$\underline{\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}} \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho\left(\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{d_x\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{d_y\}\right)\right\}$$

$$\underline{\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}} \Leftrightarrow \mathcal{F}^{-1}\left\{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\left(\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{v_1\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{v_2\}\right)\right\}$$

ADMM for Image Deconvolution with TV

<u>$\boldsymbol{x}$ – update:</u>

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{Cx} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{Dx} - \boldsymbol{v}\|_2^2$$

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)$$

- Efficient $\boldsymbol{x}$–update operates purely on 2D images with FFTs and element-wise multiplications and divisions:

$$\mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{z}) = \mathcal{F}^{-1}\left\{\frac{\boxed{\left(\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\}\right.} + \rho\left(\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{v_1\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{v_2\}\right))}{\boxed{\left(\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho\left(\mathcal{F}\{d_x\}^* \cdot \mathcal{F}\{d_x\} + \mathcal{F}\{d_y\}^* \cdot \mathcal{F}\{d_y\}\right)\right)}}\right\}$$

can pre-compute most parts

$$v_1 = \boldsymbol{v}(1{:}N), v_2 = \boldsymbol{v}(N+1{:}2N)$$

ADMM for Image Deconvolution with TV

$\boldsymbol{z}$ – update:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda \|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{z}\|_2^2$$

ADMM for Image Deconvolution with TV

$\boldsymbol{z}$ – update:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda \|\boldsymbol{z}\|_1 + \frac{\rho}{2} \|\boldsymbol{v} - \boldsymbol{z}\|_2^2$$

- Efficient $\boldsymbol{z}$–update uses element-wise soft thresholding operator $\mathcal{S}_{\kappa}(\cdot)$:

ADMM for Image Deconvolution with TV

$\boldsymbol{z}$ – update:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda \|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2$$

- Efficient $\boldsymbol{z}$–update uses element-wise soft thresholding operator $\mathcal{S}_\kappa(\cdot)$:

$$\mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \mathcal{S}_\kappa(\boldsymbol{v}) = \begin{cases} v - \kappa & v > \kappa \\ 0 & |v| \le \kappa \\ v + \kappa & v < -\kappa \end{cases} = (v - \kappa)_+ - (-v - \kappa)_+$$

$\kappa = \lambda/\rho$

This element-wise soft thresholding is the proximal operator for anisotropic TV, see course notes on block soft thresholding for isotropic TV.

$\boldsymbol{v} = \boldsymbol{D}\boldsymbol{x} + \boldsymbol{u}$

ADMM for Image Deconvolution with Denoiser

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{K}\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{u}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{u}\|_2^2$$

<u>$\boldsymbol{x}$ – update:</u>

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{x} - \boldsymbol{v}\|_2^2 \qquad \boldsymbol{v} \in \mathbb{R}^N$$

ADMM for Image Deconvolution with Denoiser

$\boldsymbol{x}$ – update:

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x}-\boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{x}-\boldsymbol{v}\|_2^2 \qquad \boldsymbol{v} \in \mathbb{R}^N$$

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{I}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{v}\right)$$

no matrix $\boldsymbol{D}$!

ADMM for Image Deconvolution with Denoiser

<u>$\boldsymbol{x}$ – update:</u>

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{x} - \boldsymbol{v}\|_2^2 \qquad \boldsymbol{v} \in \mathbb{R}^N$$

$$\boldsymbol{x} \leftarrow \left(\boldsymbol{C}^T\boldsymbol{C} + \rho\boldsymbol{I}\right)^{-1}\left(\boldsymbol{C}^T\boldsymbol{b} + \rho\boldsymbol{v}\right)$$

no matrix $\boldsymbol{D}$!

- Efficient $\boldsymbol{x}$–update operates purely on 2D images with FFTs and element-wise multiplications and divisions:

$$\mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\}$$

ADMM for Image Deconvolution with Denoiser

<u>$\boldsymbol{z}$ – update:</u>

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{z}\|_2^2 \quad \boldsymbol{v} = \boldsymbol{x} + \boldsymbol{u}$$

ADMM for Image Deconvolution with Denoiser

$\boldsymbol{z}$ – update:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2 \quad \boldsymbol{v} = \boldsymbol{x} + \boldsymbol{u}$$

$$= \arg\min_{\boldsymbol{z}} \Psi(\boldsymbol{z}) + \frac{\rho}{2\lambda}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2$$

This is a denoising problem with a regularizer that imposes a prior!

ADMM for Image Deconvolution with Denoiser

$\boldsymbol{z}$ – update:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2 \quad \boldsymbol{v} = \boldsymbol{x} + \boldsymbol{u}$$

$$= \arg\min_{\boldsymbol{z}} \Psi(\boldsymbol{z}) + \frac{\rho}{2\lambda}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2$$

- Efficient $\boldsymbol{z}$–update uses arbitrary denoiser $\mathcal{D}(\cdot)$, such as DnCNN and non-local means, using noise variance $\sigma^2 = \frac{\lambda}{\rho}$

$$\mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{x}) = \mathcal{D}\left(\boldsymbol{x}, \sigma^2 = \frac{\lambda}{\rho}\right)$$

Image Deconvolution with ADMM

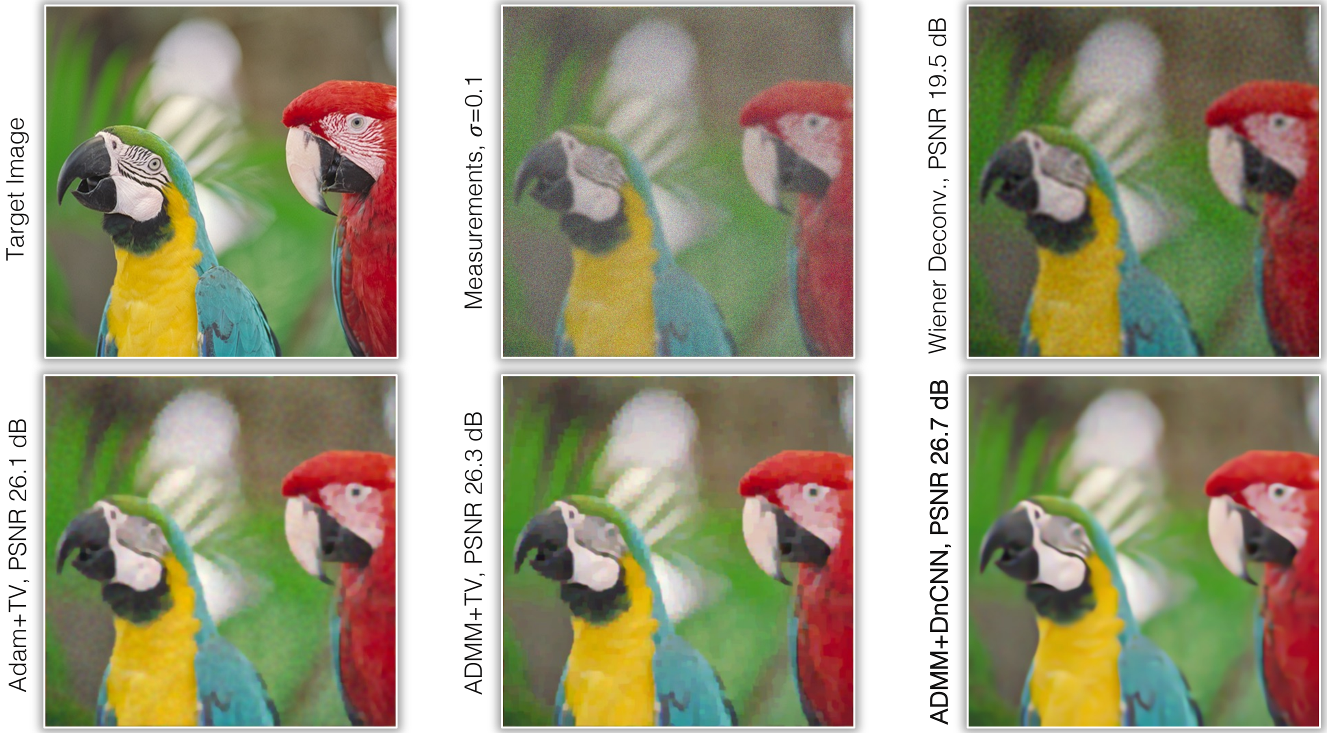

Image Deconvolution with ADMM

ADMM for deconvolution with TV

initialize ρ and λ
$x = zeros\,(W, H)\,;$
$z = zeros\,(W, H)\,;$
$u = zeros\,(W, H)\,;$
for $k = 1$ **to** max_iters **do**
$\quad v = z - u$
$\quad x = \mathbf{prox}_{\|\cdot\|_2,\rho}\,(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^*\cdot\mathcal{F}\{b\}+\rho(\mathcal{F}\{d_x\}^*\cdot\mathcal{F}\{v_1\}+\mathcal{F}\{d_y\}^*\cdot\mathcal{F}\{v_2\})}{\mathcal{F}\{c\}^*\cdot\mathcal{F}\{c\}+\rho(\mathcal{F}\{d_x\}^*\cdot\mathcal{F}\{d_x\}+\mathcal{F}\{d_y\}^*\cdot\mathcal{F}\{d_y\})}\right\}$
$\quad v = \mathbf{Dx} + \mathbf{u}$
$\quad z = \mathbf{prox}_{\|\cdot\|_1,\rho}\,(v) = \mathcal{S}_{\lambda/\rho}\,(v)$
$\quad u = u + \mathbf{Dx} - z$
end for

ADMM for deconvolution with denoiser

initialize ρ and λ
$x = zeros\,(W, H)\,;$
$z = zeros\,(W, H)\,;$
$u = zeros\,(W, H)\,;$
for $k = 1$ **to** max_iters **do**
$\quad v = z - u$
$\quad x = \mathbf{prox}_{\|\cdot\|_2,\rho}\,(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^*\cdot\mathcal{F}\{b\}+\rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^*\cdot\mathcal{F}\{c\}+\rho}\right\}$
$\quad v = x + u$
$\quad z = \mathbf{prox}_{\mathcal{D},\rho}\,(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right)$
$\quad u = u + x - z$
end for

ADMM - Convergence Criterion

- Run or "unroll" ADMM for *K* iterations
- Run until change in residual between iterations is < threshold

$$
\begin{aligned}
&v = z - u \\
&x = \mathbf{prox}_{\|\cdot\|_2,\rho}(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\} \\
&v = x + u \\
&z = \mathbf{prox}_{\mathcal{D},\rho}(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right) \\
&u = u + x - z
\end{aligned}
$$

$$
\begin{aligned}
&v = z - u \\
&x = \mathbf{prox}_{\|\cdot\|_2,\rho}(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\} \\
&v = x + u \\
&z = \mathbf{prox}_{\mathcal{D},\rho}(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right) \\
&u = u + x - z
\end{aligned}
$$

$$
\begin{aligned}
&v = z - u \\
&x = \mathbf{prox}_{\|\cdot\|_2,\rho}(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\} \\
&v = x + u \\
&z = \mathbf{prox}_{\mathcal{D},\rho}(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right) \\
&u = u + x - z
\end{aligned}
$$

$$\vdots$$

Outlook on Unrolled Optimization

- Run or "unroll" ADMM for *K* iterations
- Interpret as unrolled feedforward network:

$$
\begin{aligned}
v &= z - u \\
x &= \mathbf{prox}_{\|\cdot\|_2,\rho}(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\} \\
v &= x + u \\
z &= \mathbf{prox}_{\mathcal{D},\rho}(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right) \\
u &= u + x - z
\end{aligned}
$$

$$
\begin{aligned}
v &= z - u \\
x &= \mathbf{prox}_{\|\cdot\|_2,\rho}(v) = \mathcal{F}^{-1}\left\{\frac{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{b\} + \rho\mathcal{F}\{v\}}{\mathcal{F}\{c\}^* \cdot \mathcal{F}\{c\} + \rho}\right\} \\
v &= x + u \\
z &= \mathbf{prox}_{\mathcal{D},\rho}(v) = \mathcal{D}\left(v, \sigma^2 = \frac{\lambda}{\rho}\right) \\
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$$
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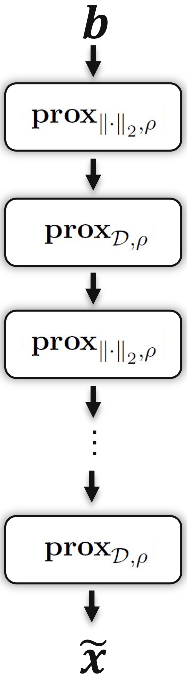
\end{aligned}
$$

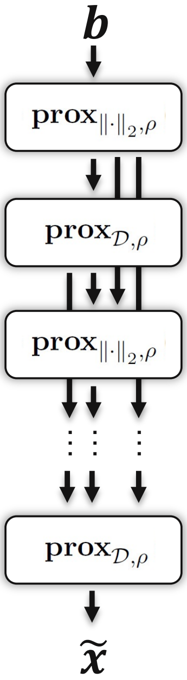
$$\vdots$$



[Diamond et al. 2017]

Outlook on Unrolled Optimization

- Run or “unroll” ADMM for K iterations
- Interpret as unrolled feedforward network:



Benefits over unrolled optimization

- Learnable parameters: $\lambda^{(k)}, \rho^{(k)}$, denoiser $\mathcal{D}^{(k)}$
- DenseNet-like skip connections
- Denoiser/regularizer can adapt to matrix $\boldsymbol{C}$
- Can train with advanced loss functions (perceptual, adversarial, other network, …)

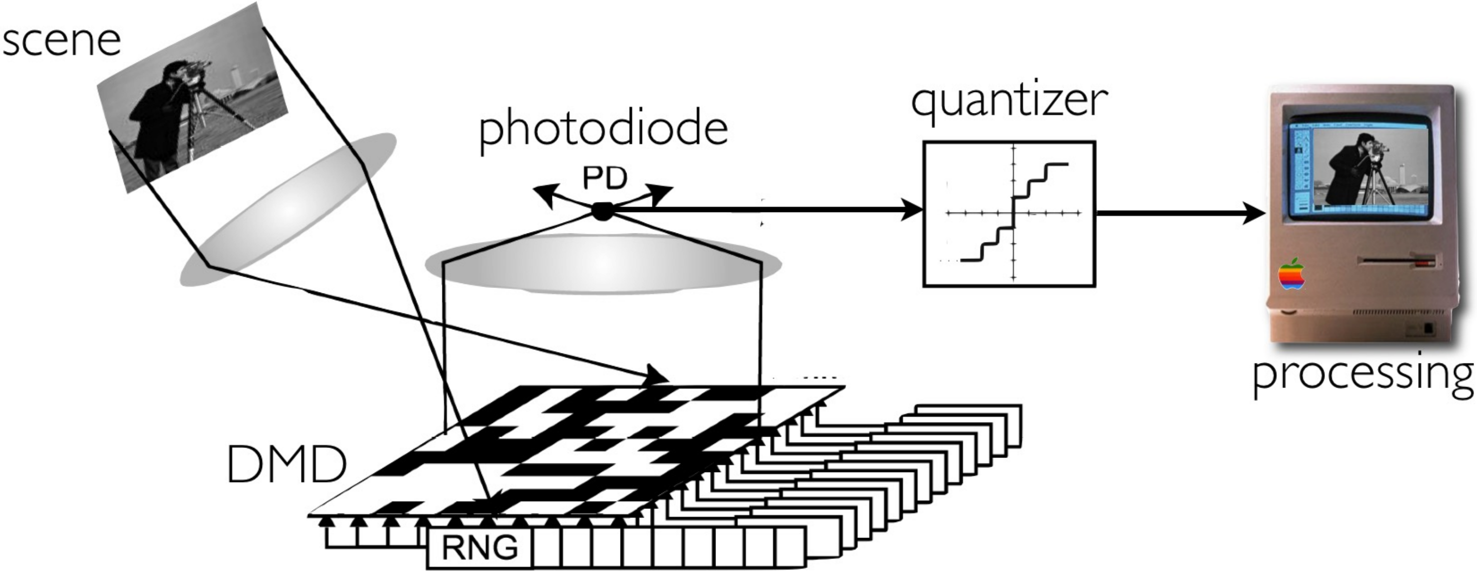
[Diamond et al. 2017]

Overview

- Brief review of deconvolution with inverse/Wiener filtering
- A Bayesian perspective of inverse problems
- Image priors/regularization and total variation
- The Alternating Direction Method of Multipliers (ADMM)
- Image deconvolution with ADMM
- Compressive imaging

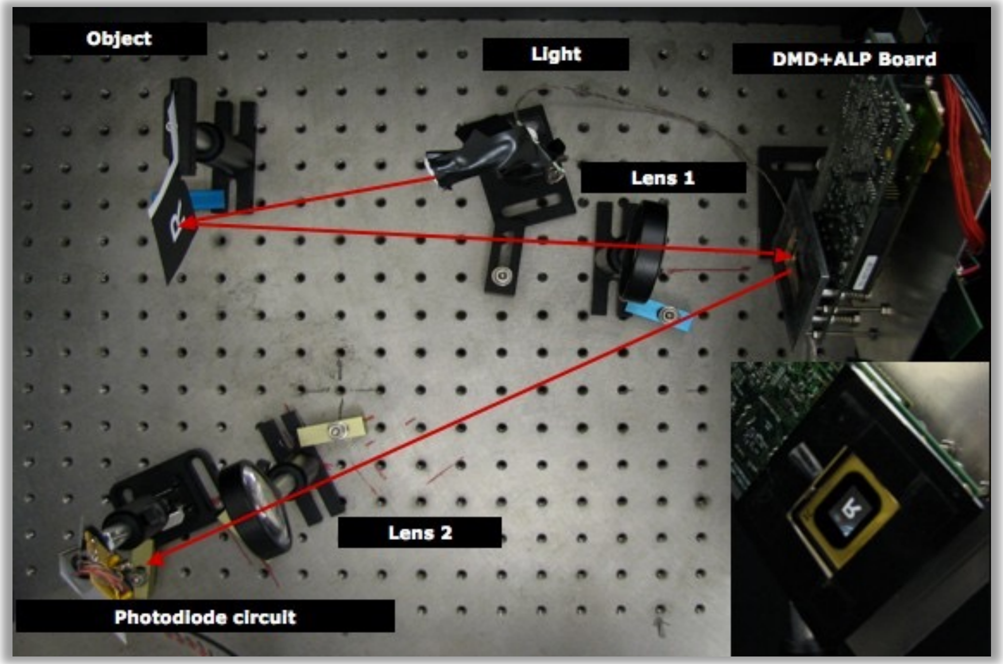
Compressive Imaging

Single-pixel Imaging



Duarte et al. 2008

Single-pixel Imaging



Duarte et al. 2008

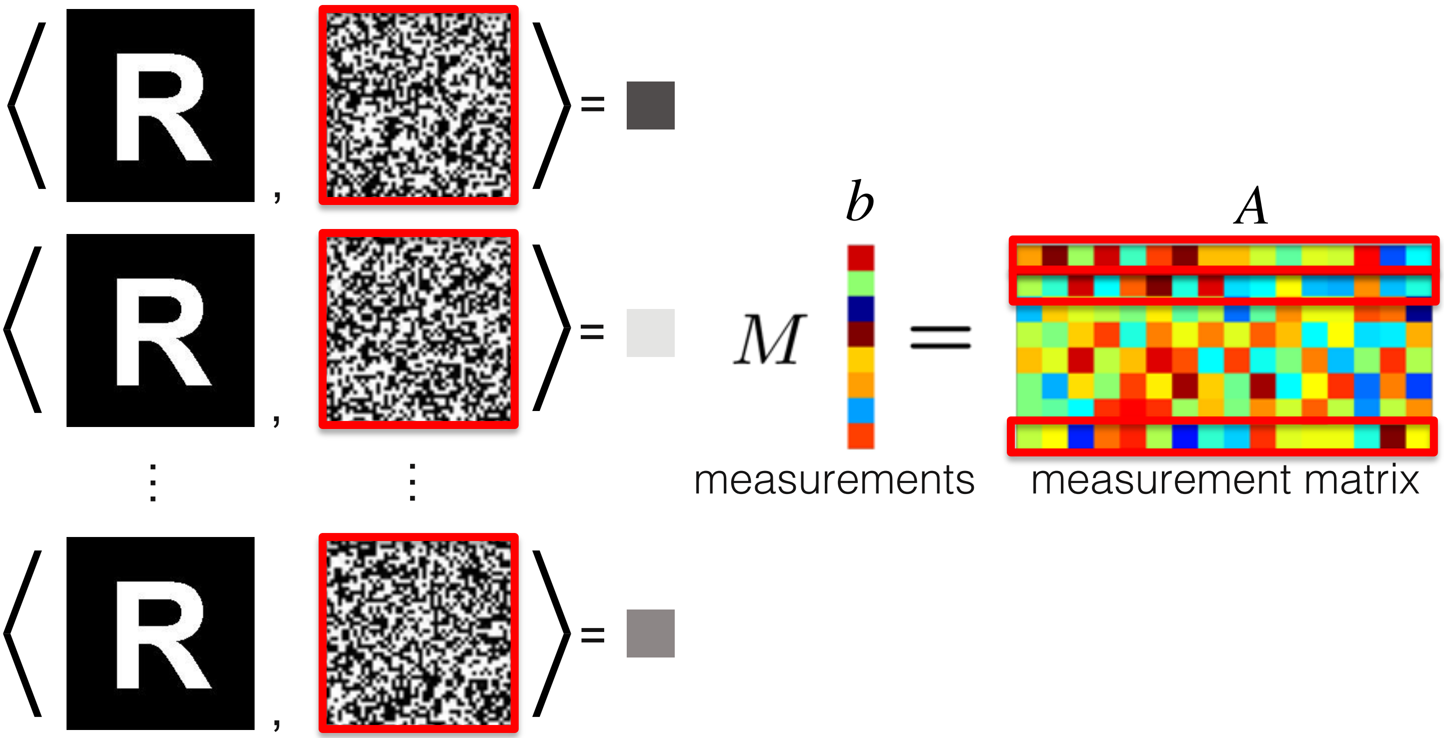
original

10%

5%

2%

Single-pixel Imaging



Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^{M}, \boldsymbol{x} \in \mathbb{R}^{N}, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- What makes it under-determined (or a compressive imaging problem):

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- What makes it under-determined (or a compressive imaging problem): $M < N$

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- What makes it under-determined (or a compressive imaging problem): $M < N$

- Problem: infinitely many solutions satisfy the observations!

  Same problem as ill-posed problems! → need **image priors**

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^{M}, \boldsymbol{x} \in \mathbb{R}^{N}, \boldsymbol{A} \in \mathbb{R}^{M\times N}$
- Standard approach – the least-norm solution: $\tilde{\boldsymbol{x}}_{\mathrm{ln}} = \boldsymbol{A}^{\boldsymbol{T}}\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{T}}\right)\boldsymbol{b}$
- This is the solution of optimization problem $$\begin{aligned} &\mathrm{minimize}_{x}\ \|\boldsymbol{x}\|_2 \\ &\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

Note: among the infinitely many solutions satisfying the observations, the least-norm solution is the one with the smallest L2 norm, thus equivalent to $\|\cdot\|_2$ regularizer

Under-determined Inverse Problems

- Image formation model: $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\eta}, \quad \boldsymbol{b} \in \mathbb{R}^M, \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{A} \in \mathbb{R}^{M \times N}$

- Standard approach – the least-norm solution: $\tilde{\boldsymbol{x}}_{\mathrm{ln}} = \boldsymbol{A}^T\left(\boldsymbol{A}\boldsymbol{A}^T\right)\boldsymbol{b}$

- Results (not great):

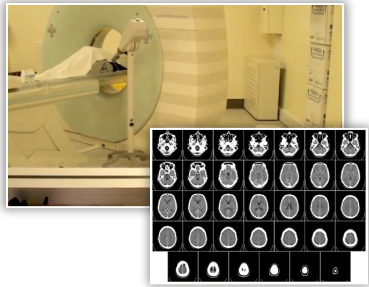
Compression Factor ${}^{N}/_{M}$

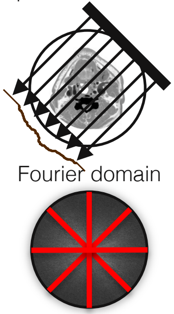
2x 4x 8x

PSNR 12.3 PSNR 10.4 PSNR 9.7

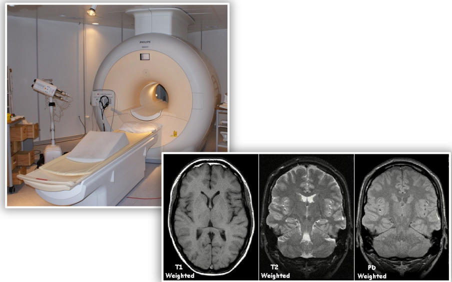
Other Inverse Problems in Imaging

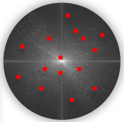
Images: Wikipedia



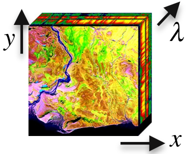


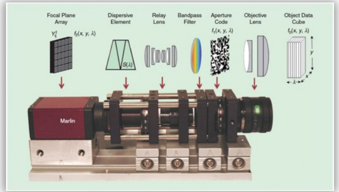
Computed Tomography (CT)





Magnetic Resonance Imaging (MRI)





Hyperspectral Imaging

- Computational photography
- Light field imaging
- Thermal imaging
- …

Other Inverse Problems in Imaging

- All these inverse problems have important applications and are very different
- Yet, they all boil down to the same inverse problem, each with a different matrix $\boldsymbol{A}$: $$\text{minimize}_{\boldsymbol{x}} \ \frac{1}{2}||\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}||_2^2 + \lambda\Psi(\boldsymbol{x})$$
- The methods derived here also apply to all those problems and applications; single-pixel imaging is a great example problem → "if you can solve this, you can solve anything"

Review of ADMM for General Inverse Problems

- Objective or "loss" function of general inverse problem:

$$\text{minimize}_{\boldsymbol{x}} \ \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \underset{\substack{\uparrow \\ \text{weight of regularizer}}}{\lambda}\Psi(\boldsymbol{x})$$

Review of ADMM for General Inverse Problems

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- Reformulate as:

$$\text{minimize}_{\{x,z\}} \ \underbrace{\frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\lambda\Psi(\boldsymbol{z})}_{g(\boldsymbol{z})}$$

$$\text{subject to } \ \boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} = 0$$

Review of ADMM for General Inverse Problems

- Objective or "loss" function of general inverse problem:

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weight of regularizer

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$$\text{subject to } \ \boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}=0$$

- Remove constraints using augmented Lagrangian

$$L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_2^2$$

Review of ADMM for General Inverse Problems

$$L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{Dx}-\boldsymbol{z}+\boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_2^2$$

- Alternating gradient descent approach to solving penalty formulation leads to following iterative algorithm:

```
while not converged:
```

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{f,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x}) + \frac{\rho}{2}\|\boldsymbol{Dx}-\boldsymbol{v}\|_2^2$$

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{g,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} L_\rho(\boldsymbol{x},\boldsymbol{z},\boldsymbol{u}) = \arg\min_{\boldsymbol{z}} g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{v}-\boldsymbol{z}\|_2^2$$

Review of ADMM for General Inverse Problems

<u>$\boldsymbol{x}$ – update:</u>

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{\|\cdot\|_2,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{v}\|_2^2$$

$$\boldsymbol{x} \leftarrow \underbrace{\left(\boldsymbol{A}^T\boldsymbol{A} + \rho\boldsymbol{D}^T\boldsymbol{D}\right)}_{\widetilde{\boldsymbol{A}}}{}^{-1}\underbrace{\left(\boldsymbol{A}^T\boldsymbol{b} + \rho\boldsymbol{D}^T\boldsymbol{v}\right)}_{\widetilde{\boldsymbol{b}}}$$

- For general inverse problems, we don't necessarily have an efficient closed-form solution for this problem, like we did for the deconvolution problem

- Use matrix-free iterative solver, such as the conjugate gradient method, to solve $\widetilde{\boldsymbol{A}}\boldsymbol{x} = \widetilde{\boldsymbol{b}}$ (e.g., `scipy.sparse.linalg.cg`)

Review of ADMM for General Inverse Problems

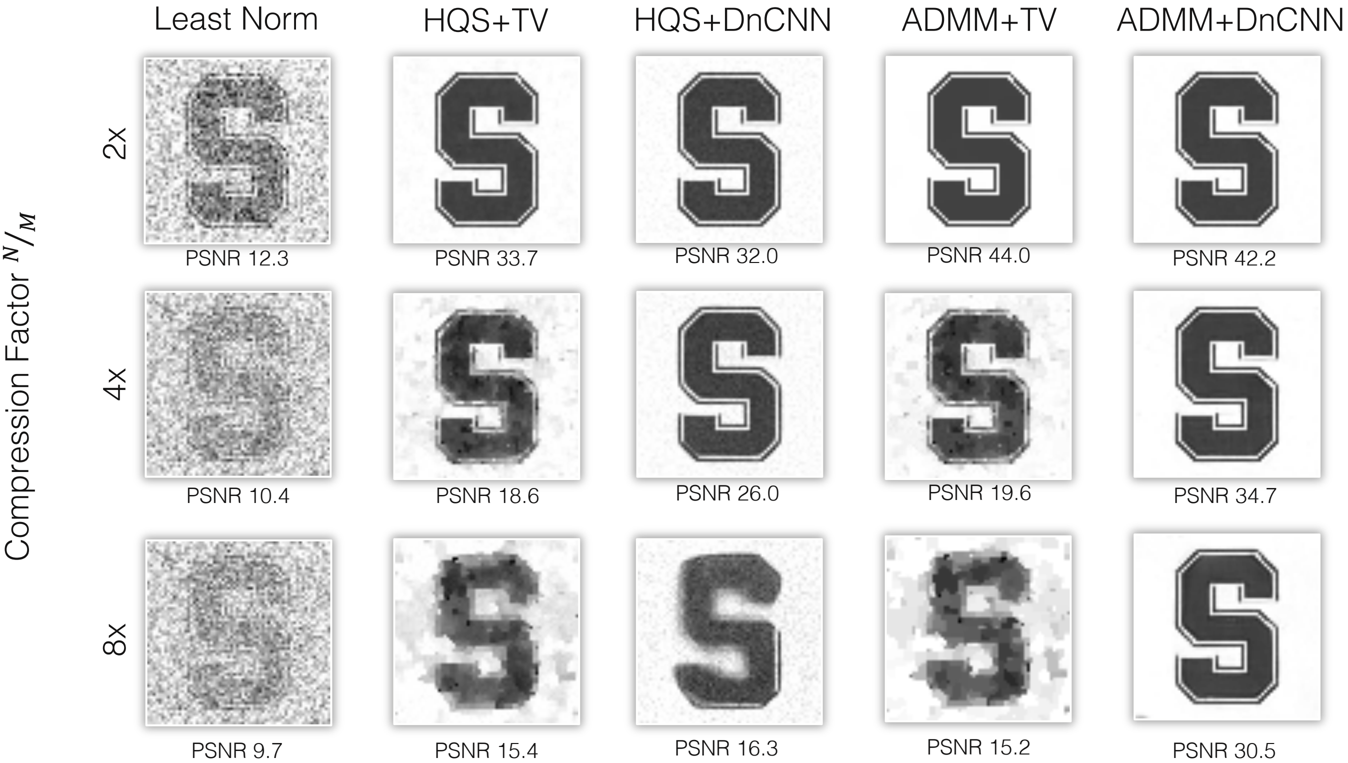
$\boldsymbol{z}$ – update for TV regularizer in closed form:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\|\cdot\|_1,\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda\|\boldsymbol{z}\|_1 + \frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{z}\|_2^2 = \mathcal{S}_{\kappa}(\boldsymbol{v})$$

$\boldsymbol{z}$ – update for denoising-based regularizer in closed form:

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{\mathcal{D},\rho}(\boldsymbol{v}) = \arg\min_{\boldsymbol{z}} \lambda\Psi(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{v} - \boldsymbol{z}\|_2^2 = \mathcal{D}\left(\boldsymbol{v}, \sigma^2 = \frac{\lambda}{\rho}\right)$$

ADMM – Results



Short tangent on Half Quadratic Splitting (HQS)
(Another solver for constrained optimization problems)

The Half-quadratic Splitting (HQS) Method

- Objective or "loss" function of general inverse problem:

$$\text{minimize}_x \ \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\Psi(\boldsymbol{x})$$

weight of regularizer

The Half-quadratic Splitting (HQS) Method

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$$\text{minimize}_x \ \frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2 + \lambda\Psi(\boldsymbol{x})$$

weight of regularizer

- Reformulate as:

$$\text{minimize}_{\{x,z\}} \ \underbrace{\frac{1}{2}\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\lambda\Psi(\boldsymbol{z})}_{g(\boldsymbol{z})}$$

$$\text{subject to} \ \ \boldsymbol{D}\boldsymbol{x} - \boldsymbol{z} = 0$$

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- Objective or "loss" function of general inverse problem:

$$\text{minimize}_x \ \frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2 + \underbrace{\lambda}_{\text{weight of regularizer}}\Psi(\boldsymbol{x})$$

- Reformulate as:

$$\text{minimize}_{\{x,z\}} \ \underbrace{\frac{1}{2}\|\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x}\|_2^2}_{f(\boldsymbol{x})} + \underbrace{\lambda\Psi(\boldsymbol{z})}_{g(\boldsymbol{z})}$$

$$\text{subject to} \ \ \boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}=0$$

- Remove constraints using penalty term (equivalent for large ρ):

$$L_\rho(\boldsymbol{x},\boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \underbrace{\frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2}_{\text{penalty term}}$$

The Half-quadratic Splitting (HQS) Method

$$L_\rho(\boldsymbol{x},\boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

- Alternating gradient descent approach to solving penalty formulation leads to following iterative algorithm:

```
while not converged:
```

$$\boldsymbol{x} \leftarrow \mathrm{prox}_{f,\rho}(\boldsymbol{z}) = \arg\min_x L_\rho(\boldsymbol{x},\boldsymbol{z}) = \arg\min_x f(\boldsymbol{x}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

$$\boldsymbol{z} \leftarrow \mathrm{prox}_{g,\rho}(\boldsymbol{D}\boldsymbol{x}) = \arg\min_z L_\rho(\boldsymbol{x},\boldsymbol{z}) = \arg\min_z g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x}-\boldsymbol{z}\|_2^2$$

The Half-quadratic Splitting (HQS) Method

$$L_\rho(\boldsymbol{x}, \boldsymbol{z}) = f(\boldsymbol{x}) + g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2$$

- Alternating gradient descent approach to solving penalty formu

- Steps not tied together with dual variable

- Can be very sensitive to the penalty parameter, requiring more tuning than ADMM (technically, penalty needs to go to infinity)

```
while n
```

$\boldsymbol{x} \leftarrow$ p ... $- \boldsymbol{z}\|_2^2$

$$\boldsymbol{z} \leftarrow \operatorname{prox}_{g,\rho}(\boldsymbol{D}\boldsymbol{x}) = \arg\min_z L_\rho(\boldsymbol{x}, \boldsymbol{z}) = \arg\min_z g(\boldsymbol{z}) + \frac{\rho}{2}\|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{z}\|_2^2$$

Applications of Compressive Imaging

Compressive Medical Imaging

- reduce acquisition time, radiation exposure, or allow for more patients in same time, …

- examples: x-ray computed tomography and MRI

Computed Tomography (CT)

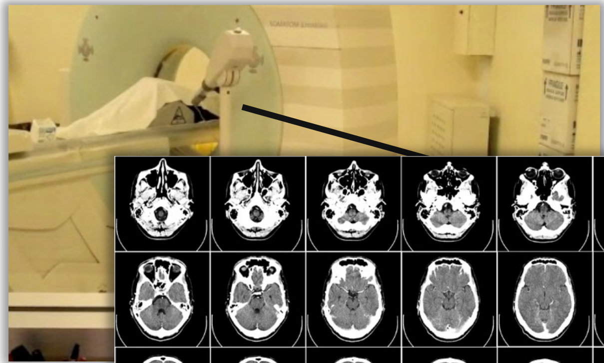
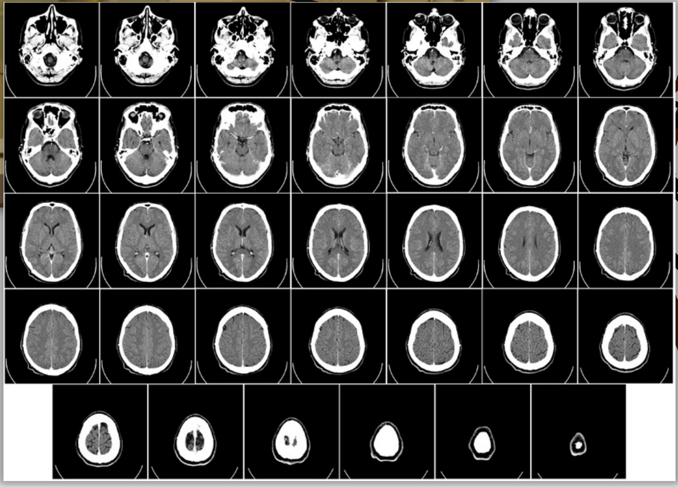
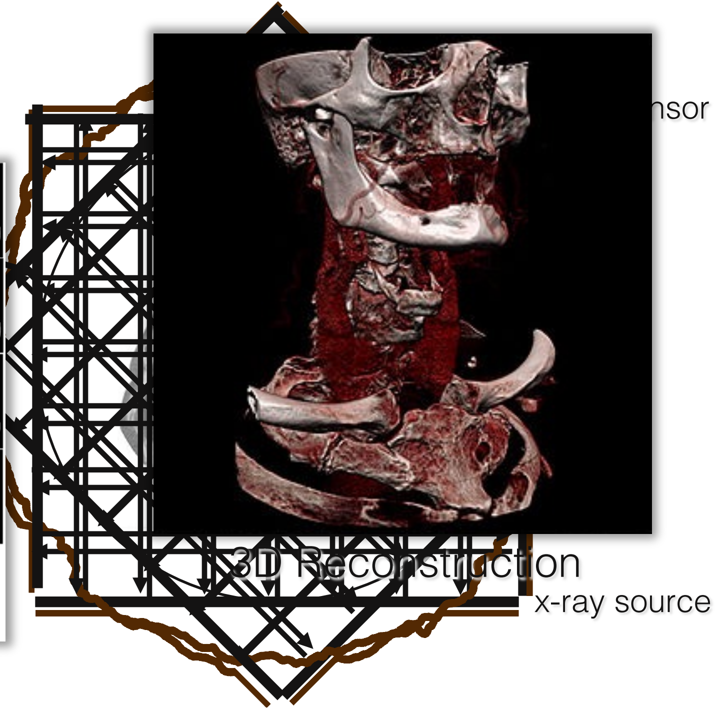


Image: Wikipedia



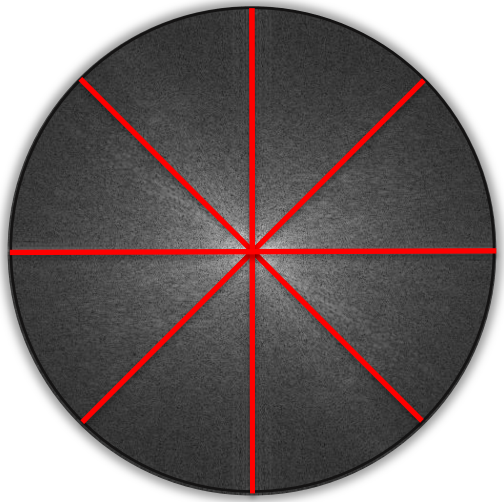
Reconstructed 2D Slices



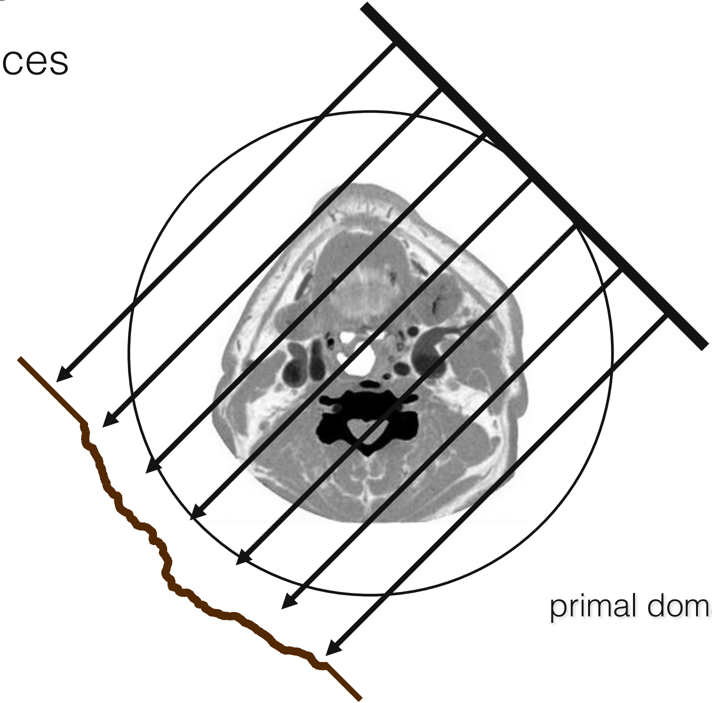
3D Reconstruction

Computed Tomography – Fourier Slice Theorem

- measurements = Fourier slices
- compressive CT: e.g. fewer slices



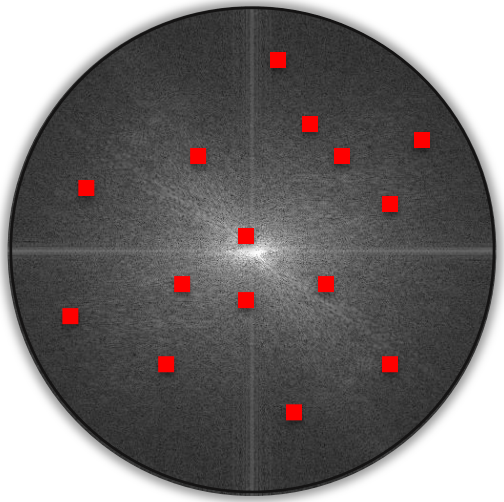
frequency domain



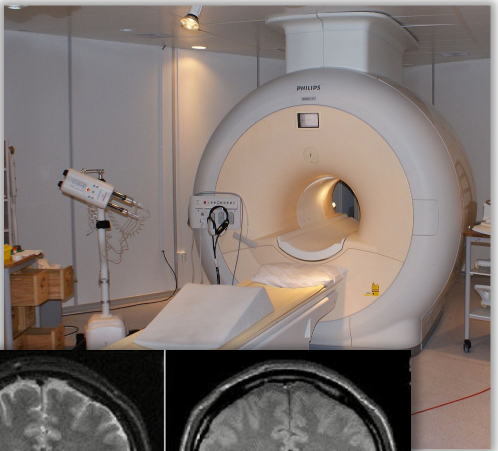
primal domain

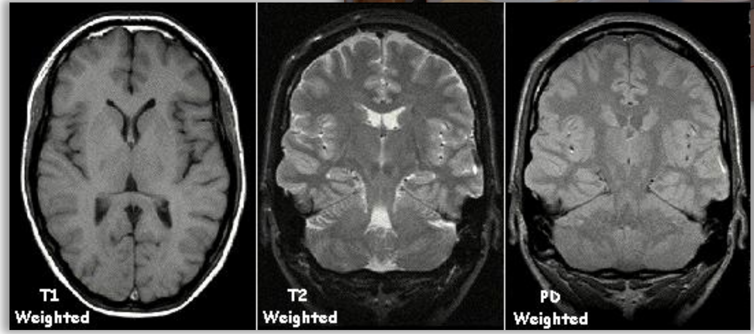
Magnetic Resonance Imaging

- measurements = (random) Fourier coefficients
- compressive MRI: fewer Fourier coefficients



frequency domain





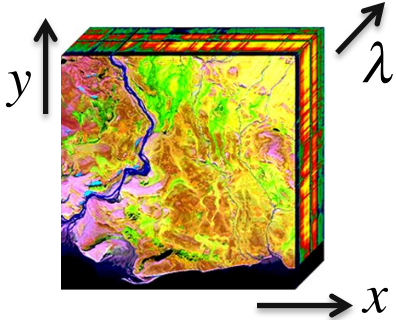
wikipedia

Compressive Imaging: CT & MRI

- people in bio-medical imaging often hesitant about priors:
  - few guarantees for success
  - if reconstruction breaks, not clear how exactly
  - is that feature a reconstruction artifact or the thing I’m looking for?

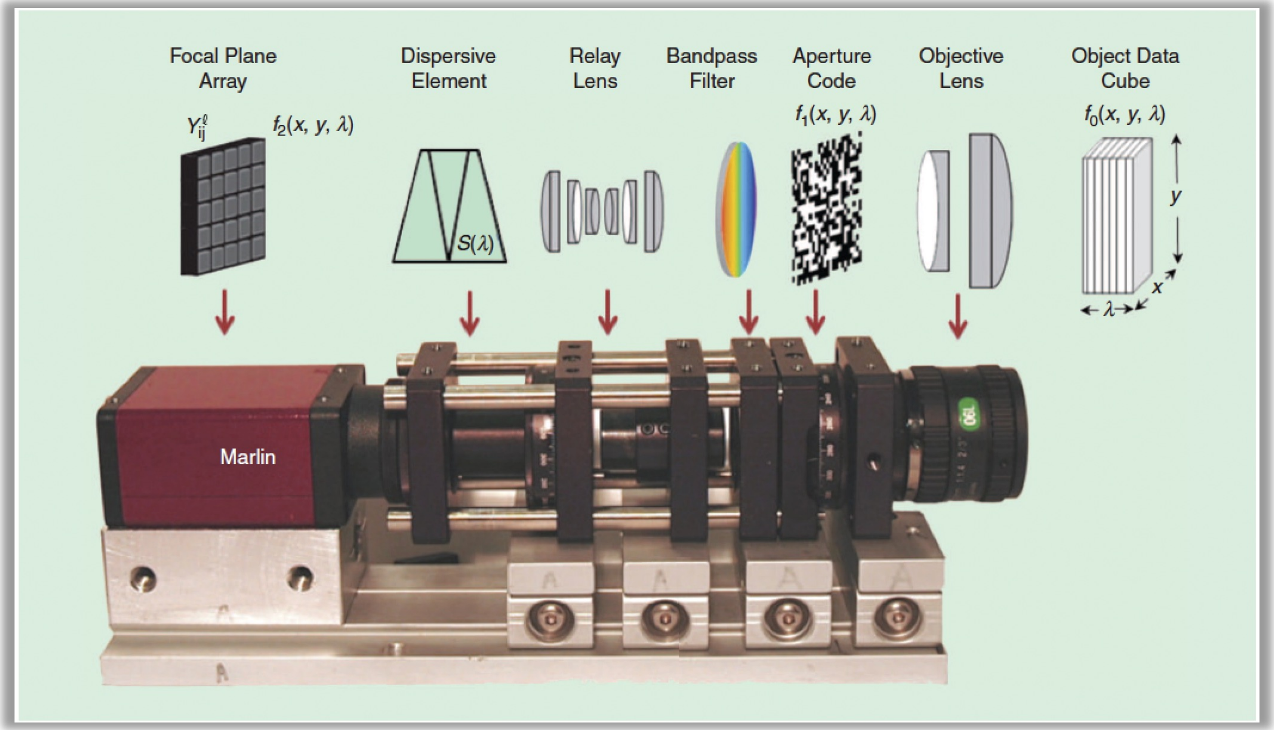
Compressive Hyperspectral Imaging

- motivation:



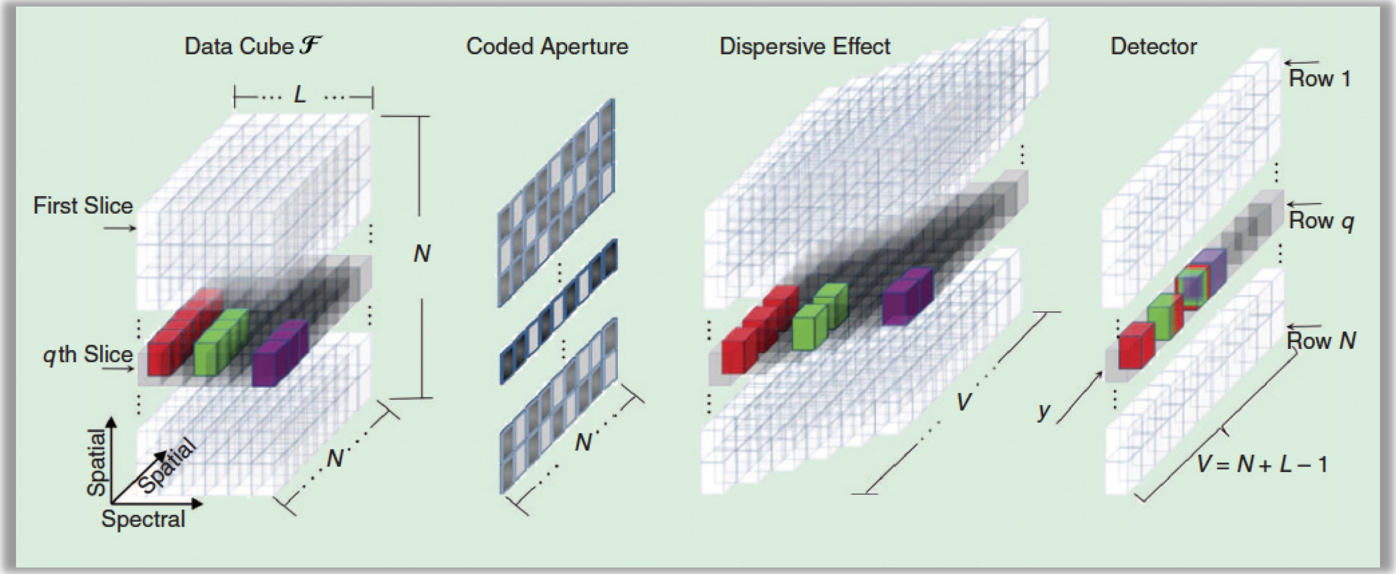
- conventional: either scan over xy or over lambda!
- idea: capture hyperspectral datacube with a single, coded image – use compressive sensing to reconstruct
- first approach: CASSI (coded aperture snapshot spectral imager), Wagadarikar 2008

Compressive Hyperspectral Imaging



Arce et al. 2014

Compressive Hyperspectral Imaging



Arce et al. 2014

Compressive Hyperspectral Imaging

- moderate quality for snapshot, but good quality for coded multi-shot
- applications: remote sensing, cultural heritage, …

Arce et al. 2014

Compressive Imaging Everywhere

- metamaterials
- THz imaging
- x-ray imaging
- thermal IR
- ultra-fast imaging
- not as much on compressive coherent imaging (could be interesting for course projects: OCT, holography, …)
- …

Notes

- compressive imaging is an exploding area: check COSI, ICCP, CVPR, ICCV conferences, other optics journals and conferences
- most variants of compressive imaging problems can be implemented with ADMM
- check lecture notes online to help with homework
- Increasingly we want to learn the sensing matrices, reconstruction using neural networks and datasets…

References and Further Reading

Must read: course notes on Image Deconvolution with ADMM & course notes on compressive imaging

Adam

- D. Kingma, J. Ba “Adam: A method for stochastic optimization”, ICLR 2015

ADMM

- S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein “Distributed optimization and statistical learning via the alternating direction method of multipliers”, Foundation and Trends in Machine Learning, 2001

Single-pixel Imaging

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Unrolled Optimization

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