Nonlinear Equations (Chapter 5)

Problem: find \( x \) such that \( f(x) = y \)

where \( f \) is any smooth function.

(for now, \( f : \mathbb{R} \to \mathbb{R} \))

Equivalently: find the root(s) of a general equation of the form \( f(x) = 0 \)

i.e. find an \( x^* \) such that \( f(x^*) = 0 \).

Existence and Uniqueness

Unlike in a linear system \( f(x) = 0 \) can have:

- no solution  \( \text{eg. } \ e^x + 1 = 0 \)
- one solution  \( \text{eg. } \ e^x - x = 0 \)
- many solutions  \( \text{eg. } \ x^3 + 6x^2 + 11x - 6 = 0 \)
- inf. many solutions  \( \sin(x) - 1 = 0 \)

Even though we can't make any global assertions about existence / uniqueness of roots, there are useful local criteria that guarantee the existence of solutions.

Intermediate Value Thm

If \( f \) is continuous on an interval \([a, b]\), and \( \nu \) is a number between \( f(a) \) and \( f(b) \), then there exists \( c \in [a, b] \) such that \( f(c) = \nu \).
We can apply I.V.T. to root finding:

If \( f \) is continuous, \( f(L) \) and \( f(R) \) have opposite signs, then there is a root \( x^* \) between \( L \) and \( R \) with \( f(x^*) = 0 \).

We can use this idea to develop an algorithm for root finding:

\[ \text{eg 1: } f(x) = \cos(x) - x \]

\[ f(0) = 1 \quad f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} \]

\( \Rightarrow \) There should be a root \( f(x^*) \)

\( x^* \in \left[0, \frac{\pi}{2}\right] \)

Consider the midpoint \( m_1 = 0 + \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4} \)

we evaluate \( f(m_1) = -0.07829 \). \( \in \) negative.

Reapply the I.V.T. \( \Rightarrow \) There should be a root \( f(x^*) \)

\[ f(0) > 0 \]
\[ f(m_1) < 0 \]

\( x^* \in \left[0, \frac{\pi}{4}\right] \)
Consider the next midpoint \( m_2 = 0 + \frac{\pi}{4} - 0 = \frac{\pi}{8} \).

\[ f(m_2) = 0.53118 \quad \text{positive} \]

Reapply the IVT \( \Rightarrow \). There should be a root \( f(x^*) \)

\[ f(m_2) > 0 \]
\[ f(m_1) < 0 \]

\[ \Rightarrow x^* \in \left[ \frac{\pi}{8}, \frac{\pi}{4} \right] \]

\[ m_2 \quad \text{or} \quad m_1 \]

Algorithm: Interval Bisection.

We start with \( f \) continuous, and values \( a, b \) such that \( \text{sign}(f(a)) \neq \text{sign}(f(b)) \).

Iteratively find \( m = a + \frac{b-a}{2} \).

1. \( \text{sign}(f(a)) = \text{sign}(f(m)) \).
   \[ a := m \quad \text{new interval: } [m, b] \]
2. \( \text{else} \):
   \[ b = m, \quad \text{new interval: } [a, m] \]

Question:
* When do we stop?
* How quickly does Interval Bisection Converge?
* Can we do better? (Yes).
* What do we say about the conditioning of the root-finding problem \( f(x) = 0 \).
**Stopping Criteria**

**Idea #1:** What about when \( f(m) \) is small enough. Can be fooled by flat functions:

\[
\text{\( f(m) \) is small} \quad \implies \quad |m - x^*| \text{ large (absolute error)}
\]

**Idea #2:** What about when the interval size is sufficiently small? \( \Rightarrow \) fixed number of iterations:

\[
|a_i - b_i| \leq \varepsilon
\]

Can be fooled by functions like this:

\[
|x^*-m| \text{ is small} \quad \implies \quad f(m) \text{ is large}
\]

In practice, the stopping criteria is (real life) problem dependent.

**Convergence Rate**

We use notation \( e_k \) to denote the absolute error

\[
e_k = x_k - x^* \quad \text{at iteration} \ k
\]

where \( x_k \) is the approx. solution at iteration \( k \).

\( x^* \) is the true solution.
In interval bisection, we will approximate \( e_k = |b_k - a_k| \).

An iterative method converges with rate \( r \) if

\[
\lim_{k \to \infty} \frac{||e_{k+1}||}{||e_k||} = C
\]

for some constant \( C > 0 \).

In particular if

- \( r = 1 \), \( C < 1 \). Convergence rate is \( \text{linear} \).
- \( r > 1 \).
- \( r = 2 \).
- \( r = 3 \).

For interval bisection,

\[
|e_k| = |b_k - a_k|
\]

\[
= \frac{|b-a|}{2^k}.
\]

\[
\therefore \lim_{k \to \infty} \frac{|b-a|/2^{k+1}}{|b-a|/2^k} = \lim_{k \to \infty} \frac{1}{2} = \frac{1}{2}
\]

\[\Rightarrow\text{ Interval Bisection converges linearly.}\]

with \( r = 1, \ C = \frac{1}{2} \).

\[\Rightarrow\text{ we gain one additional correct bit of information per iteration.}\]
Consider $F(\beta=10, \ \rho=2, \ \mu=10, \ \lambda=-10)$.

\[
a = 0.67 = 6.7 \times 10^{-1} \\
b = 0.69 = 6.9 \times 10^{-1}.
\]

\[
m = \frac{a + b}{2} \quad \quad a + b = 1.36 \Rightarrow 1.4 \times 10^0.
\]

\[
\frac{F(a+b)}{2} = 0.70 \Rightarrow 7.0 \times 10^{-1}.
\]

Problem! \quad m = 0.70 \quad \text{is not even between } [a, b]

The computation \quad m = a + \frac{b-a}{2} \quad \text{guarantees that}

m \in [a, b].

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**Fixed-Point Iteration**

Find an equivalent problem to $f(x^*) = 0$ with the same solution.

\[ \Rightarrow \] Find $p^*$ that is a fixed-point of $g(x)$:

\[ \text{i.e.} \quad g(p^*) = p^* \]

\[ \text{eg'll say we want to find a root of} \]

\[ f(x) = x - 0.2 \sin(x) - 0.5 \]
Define \( g(x) = 0.2 \sin(x) + 0.5 \).

Notice that if \( x^* = g(x^*) \) is a fixed-point of \( g \),
then \( f(x^*) = 0 \).

i.e. \( x - 0.2 \sin(x) - 0.5 = 0 \).
\( \iff x = 0.2 \sin(x) + 0.5 \)

Fixed-points of \( g(x) \) are roots of \( f(x) \).

Fixed-points are values where \( g(x) \) intersects \( y = x \).

We can find fixed-points of \( g(x) \) using functional iteration (fixed-point iteration).

Start with: \( x_0 \) given
\[ x_{i+1} = g(x_i) \quad i = 0, 1, 2, \ldots \]

If this iteration converges, it converges to a fixed point of \( g \).
Example 1: $f(x) = x - 0.2 \sin(x) - 0.5$.

Run bisection $\Rightarrow$ ~22 iters to convergence.

with $g(x) = 0.2 \sin(x) + 0.5$.

Start with $x_0 = 0$ $\Rightarrow$ ~8-9 iters.

$\cdot x_0 = 1.0$ $\Rightarrow$ ~8-9 iters

$\cdot x_0 = -1.0$ $\Rightarrow$ ~8-9 iters

$\cdot x_0 = 1000$ $\Rightarrow$ ~8-9 iters

$\cdot x_0 = -1000$ $\Rightarrow$ ~8-9 iters

$g'(x) < 1$

Example 2: find roots of $f(x) = x^3 - x - 1$.

$g(x) = x^3 - 1$.

$\cdot x^3 - x - 1 = 0$

$x^3 - 1 = x$
Try \( g(x) = (x+1)^{1/3} \).

\[ x^3 - x - 1 = 0 \]
\[ x = \sqrt[3]{x+1} \]

**Convergence of fixed-point iteration**

Suppose \( g(x) \) has a fixed point \( x^* = g(x^*) \).

If \( |g'(x^*)| < 1 \) then there is an interval containing \( x^* \) such that fixed-point iteration

\[ x_{k+1} = g(x_k) \]

converges to \( x^* \) if \( x_0 \) is in that interval.

If \( |g'(x^*)| > 1 \), then iterative scheme diverges.

The asymptotic convergence rate is usually linear with \( C = |g'(x^*)| \).

If \( g'(x^*) = 0 \), then the convergence is at least quadratic.