Midterm 3 over, yay!

Rest of today:
- Complete Pivoting
- Iterative Refinement
- Modified Problems

Complete Pivoting

In partial pivoting we choose the largest element below the diagonal to be our pivot.

eg//
$$\begin{bmatrix} 1 & 2 & 5 & 9 \\ 0 & 3 & -1 & -2 \\ 0 & 4 & 2 & 1 \\ 0 & -1 & 5 & 0 \end{bmatrix}$$

partial pivoting: $R_2 \leftrightarrow R_3$ pivot $= 4$.

full pivoting: $R_2 \leftrightarrow R_4$. $C_2 \leftrightarrow C_3$ pivot $= 5$

~~Before~~

In complete (full) pivoting, we search for the largest entry in the remaining submatrix.

Q: Does exchanging columns change our solution.

Yes. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad x_1 = \frac{1}{2}$, $x_2 = \frac{1}{3}$

But if we exchange columns

$\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}$

The solution will change, but predictably.
We need to keep track of the column swaps.

In practise, full pivoting is very expensive.

and partial pivoting is usually good enough.

Iterative Refinement

Suppose $\underline{x_0}$ is a solution to $A\underline{x} = \underline{b}$.

but we have a residual $r_0 = \underline{b} - A\underline{x_0} \neq 0$.

We can obtain a better solution by solving

$$A\underline{z_0} = \underline{r_0}$$

Then if we let $\underline{x_1} := \underline{x_0} + \underline{z_0}$

$$\begin{aligned} A\underline{x_1} &= A(\underline{x_0} + \underline{z_0}) \\ &= A\underline{x_0} + A\underline{z_0} \\ &= (\underline{b} - \underline{r_0}) + \underline{r_0} \\ &= \underline{b}. \end{aligned}$$

If $\underline{z_0}$ is not exact, we can repeat this process.

Not practical in practise for this problem

but is ~~a~~ useful idea in general.

Modified Problems.

If we have a solution to $A\underline{x} = \underline{b}$.

what if we have a different

(1) b. $\rightarrow$ $A\underline{x} = \underline{b}'$ $\rightarrow$ Keep the LU factorization of A. $LU\underline{x} = \underline{b}'$

then $\begin{cases} L\underline{y} = \underline{b}' \\ U\underline{x} = \underline{y} \end{cases}$

(2). A.

In general, if the matrix A changes, then we have an entirely different problem, and we start over.

In some cases, we can make use of the existing solution $A\underline{x} = \underline{b}$:

Rank one update $\quad A_1 = A - \underbrace{uv^T}_{\text{this update matrix has rank 1.}}$

u is $n \times 1$ vector

v is $n \times 1$ vector

In this case, we can use the Sherman-Morrison Formula to compute

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^TA^{-1}u)^{-1}v^TA^{-1}$$

Then

$$x = (A - uv^T)^{-1}b$$

$$= \underbrace{A^{-1}b}_{y} + \underbrace{A^{-1}u}_{z}(1 - v^T\underbrace{A^{-1}u}_{z})^{-1}v^T\underbrace{A^{-1}b}_{y}.$$

1. Solve $A\underline{z} = \underline{u}$
2. Solve $A\underline{y} = \underline{b}$
3. $y + z\underbrace{(1 - v^Tz)^{-1}}_{\text{scalar}}v^Ty$.

Pf. of Sherman-Morrison Formula:

we'll show that $(A-uv^T)(A^{-1}+A^{-1}u(1-v^TA^{-1}u)^{-1}v^TA^{-1}) = I$.

Let $\gamma = \underbrace{(1-v^TA^{-1}u)^{-1}}_{\text{note this is a scaler!}}$ to reduce the amount of writing

$$(A-uv^T)(A^{-1}+A^{-1}uv^TA^{-1}\gamma)$$

Expand.

$$= AA^{-1} - uv^TA^{-1} + AA^{-1}uv^TA^{-1}\gamma - uv^TA^{-1}uv^TA^{-1}\gamma$$

simplify $AA^{-1}=I$

$$= I - uv^TA^{-1} + uv^TA^{-1}\gamma - uv^TA^{-1}uv^TA^{-1}\gamma$$

factor out u at beginning $v^TA^{-1}\gamma$ at end of both terms

$$= I - uv^TA^{-1} + u\left(1-v^TA^{-1}u\right)v^TA^{-1}\gamma$$

write out γ

$$= I - uv^TA^{-1} + \frac{u\cancel{(1-v^TA^{-1}u)}v^TA^{-1}}{\cancel{(1-v^TA^{-1}u)}}$$

$$= I - uv^TA^{-1} + uv^TA^{-1}$$

$$= I.$$

$$\therefore (A-uv^T)^{-1} = \left(A^{-1} + \frac{A^{-1}uv^TA^{-1}}{1-v^TA^{-1}u}\right)$$