Last time

Solving $A\underline{x}=\underline{b}$ for $A \in \mathbb{R}^{n\times n}$ nonsingular using
{ Gauss Elimination
{ LU Factorization.

But LU factorization is not the end of the story.

① $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ pivot is zero! can't divide by zero.

$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0.00001 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ pivot almost zero

②. $A = \underbrace{\begin{bmatrix} 4.11 & 5.0 \\ 8.23 & 10.1 \end{bmatrix}}_{\text{this matrix is "almost singular"}}$ $\underline{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $A = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}$

Q: How do we analyze the condition of $A\underline{x}=\underline{b}$

③ Q How do we compare solutions $\underline{x}$ from different algo.

$A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$

Two solutions:

$\underline{x}_1 = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}$ $\underline{x}_2 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}$ but $x^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Which is better? Maybe look at the residual

$A\underline{x}=\underline{b}$ but $A\underline{x}_1 - \underline{b} =: \underline{r}_1 = \begin{bmatrix} 0.000001 \\ 0 \end{bmatrix}$

$A\underline{x}_2 - \underline{b} =: \underline{r}_2 = \begin{bmatrix} 0.001343 \\ 0.001572 \end{bmatrix}$.

The norm ("size") of the residual is a poor measure of the closeness of a solution $\|\underline{x}^* - \hat{\underline{x}}\|$.

Besides, linearly scaling A, $\underline{b}$ will affect $\underline{r}$ but not $\|\underline{x}^* - \hat{\underline{x}}\|$

To talk about all this, we need a way to talk about the "size" of a vector and also a matrix.

<u>Def</u> A <u>vector norm</u> $\|\cdot\|$ is a mapping $\mathbb{R}^n \to \mathbb{R}^{\geq 0}$ such that

1). $\forall \underline{x} \in \mathbb{R}^n \quad \|\underline{x}\| \geq 0$ with $\|\underline{x}\| = 0$ iff $\underline{x} = \underline{0}$

if and only if

2) $\forall \underline{x} \in \mathbb{R}^n, c \in \mathbb{R} \quad \|c\underline{x}\| = |c| \cdot \|x\|$

3) $\forall \underline{x}, \underline{y} \in \mathbb{R}^n \quad \|\underline{x} + \underline{y}\| \leq \|x\| + \|y\|$ — triangle inequality.

In this course we will use the p-norm:

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

1-norm: $p = 1$. $\quad \|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$

2-norm: $p = 2 \quad \|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ ← Euclidean norm.

∞-norm: $p = \infty \quad \|\underline{x}\|_\infty = \max_i |x_i|$.

<u>Def</u> An <u>induced matrix norm</u> is a matrix norm defined in terms of a vector norm as follows:

$$\underbrace{\|A\|}_{\text{matrix norm}} = \max_{\|x\|=1} \underbrace{\|Ax\|}_{\text{vector norm}} = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

← vector norms.

matrix norms satisfy:

1). $\|A\| \geq 0 \quad \|A\| = 0$ iff $A = \underline{0}$ ← matrix zero

2). $\|c \cdot A\| = |c| \cdot \|A\| \quad c \in \mathbb{R}$.

3). $\|A+B\| \le \|A\| + \|B\|$.

4) $\|A \cdot B\| \le \|A\| \cdot \|B\|$

5) $\|A\underline{x}\| \le \|A\| \, \|\underline{x}\|$

$\|I\| = 1$.

we can show that

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}| \quad \text{— max abs column sum}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \quad \text{— max abs row sum}$$

Condition Number of a Matrix

$$\text{Cond}(A) = \begin{cases} \|A\| \cdot \|A^{-1}\|. & \text{if } A \text{ is nonsingular} \\ \infty & \text{if } A \text{ is singular} \end{cases}$$

we can show that

$$\|A\| \cdot \|A^{-1}\| = \left(\max_{\|x\|=1} \|Ax\|\right)\left(\min_{\|x\|=1} \|Ax\|\right)^{-1}.$$

Cond(A) describes the ratio of the max stretch to max shrinking of points on the unit sphere.

Propecties of cond(A).

1. $\text{Cond}(A) = \|A\| \cdot \|A^{-1}\| \underset{\text{property ④ of matrix norm.}}{\ge} \|A \cdot A^{-1}\| = \|I\| = 1$

   $\text{Cond}(A) \ge 1$.

2. $\text{Cond}(I) = 1$.

3. $\text{Cond}(\gamma A) = \|\gamma A\| \cdot \|\frac{1}{\gamma} A^{-1}\| = \text{Cond}(A)$. $\gamma \in \mathbb{R}$. $\gamma \ne 0$.

Notation: $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

$\max_i |x_i|$ = the largest possible $|x_i|$ across all possible values of i.

$\max_{1 \le i \le 4} |x_i|$

eg//. For $\underline{x} = \begin{bmatrix} 2 \\ 3 \\ -4 \\ 1 \end{bmatrix}$ $\max_i |x_i| = 4$.

Notation.

$\max_{\underline{x}} \|A\underline{x}\|$ ~ the largest possible value for $\|A\underline{x}\|$ across all possible values of $\underline{x}$

$\max_{\underbrace{\|\underline{x}\|=1}_{\text{condition on } \underline{x}}} \|A\underline{x}\|$ ~ the largest possible value for $\|A\underline{x}\|$ across all possible values of $\underline{x}$ where $\|\underline{x}\| = 1$.

If $\text{cond}(A)$ is small then $A\underline{x} = \underline{b}$ is well-conditioned.

with the bound.

$$\frac{\|\Delta x\|}{\|x^*\|} \le \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}.$$

and similarly

$$\frac{\|\Delta x\|}{\|x^*\|} \le \text{cond}(A) \frac{\|\Delta A\|}{\|A\|}.$$

Deriving the condition number

Recall from lecture 1. that $C.N = \dfrac{|\Delta y / y|}{|\Delta x / x|}$

$$\underbrace{\left|\frac{\Delta y}{y}\right|}_{\substack{\text{relative} \\ \text{error of output}}} = C.N. \underbrace{\left|\frac{\Delta x}{x}\right|}_{\substack{\text{relative} \\ \text{error of input}}}.$$

Consider $A\underline{x} = \underline{b}$.

Let $\underline{x^*}$ be the true solution $A\underline{x^*} = \underline{b}$.

$\underline{\hat{x}}$ be the computed solution. $A\underline{\hat{x}} - \underline{b} = \underline{\Delta b}$ Residual

$$\underline{\Delta x} = \underline{\hat{x}} - \underline{x^*}$$

$$
\begin{array}{rl}
 & A\underline{\hat{x}} - \underline{b} = \underline{\Delta b} \\
- & A\underline{x^*} - \underline{b} = 0 \\
\hline
 & A(\underline{\hat{x}} - \underline{x^*}) = \Delta b.
\end{array}
$$

$~~A x^* = b~~$

$~~A \hat{x} - b = \Delta b~~$

$$\Rightarrow \quad A\,\Delta x = \Delta b.$$

We have: $\underline{\Delta x} = A^{-1}\underline{\Delta b} \Rightarrow \|\underline{\Delta x}\| \le \|A^{-1}\| \cdot \|\underline{\Delta b}\|$ — (1)

$A\underline{x^*} = \underline{b} \Rightarrow \|A\| \cdot \|\underline{x^*}\| \ge \|b\|.$ — (2)

Take ① / ②

$$\frac{\|\Delta x\|}{\|A\| \cdot \|x^*\|} \leq \frac{\|A^{-1}\| \cdot \|\Delta b\|}{\|b\|}$$

Compare to:

$$\left|\frac{\Delta y}{y}\right| = C.N. \left|\frac{\Delta x}{x}\right|.$$

rel. error output. — rel. error input

$$\underbrace{\frac{\|\Delta x\|}{\|x^*\|}}_{\text{norm relative error of computed solution}} \leq \underbrace{\|A\| \cdot \|A^{-1}\|}_{\text{C.N. cond}(A)} \underbrace{\frac{\|\Delta b\|}{\|b\|}}_{\text{norm relative residual of computed solution}}$$

What this means is that

if cond (A) is small — property of problem

and relative residual is small — property of the computed solution.

then: relative error is small.

Alternatively, if Δb small but cond(A) is large

then relative error could be large.

Improving stability of Gauss Elimination

eg//

$$\underbrace{\left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 1 & \end{array}\right]}_{\text{can't divide by zero}} \qquad \underbrace{\left[\begin{array}{cc|c} 0.00000 & 1 & 1 \\ 1 & & 1 \end{array}\right]}_{\text{multiplication by a large number will increase relative error.}}$$

main idea: interchange rows ($R_1 \leftrightarrow R_2$) <u>before</u> eliminating ~~a~~ the ~~...~~ elements below a diagonal. So ~~...~~ to maximize the abs. value of the <u>pivot</u> (value in the diagonal) e.g.// $A = \begin{bmatrix} 3 & 2 & 9 \\ 4 & 5 & 1 \\ -5 & 2 & 3 \end{bmatrix}$ what row ~~share exchange~~ interchange should we make before performing elimination?

$R_1 \leftrightarrow R_3$.

$$A' = \begin{bmatrix} -5 & 2 & 3 \\ 4 & 5 & 1 \\ 3 & 2 & 9 \end{bmatrix}$$

We will use permutation matrices to express the interchange

e.g.// $R_1 \leftrightarrow R_3$ $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $A' = P_1 A$.

<u>Def</u> A <u>permutation matrix</u>. P has exactly one 1 in each row and each column, and ~~zero~~ 0 everywhere else.

<u>Facts</u>:

* $P^{-1} = P^T$
* $\text{cond}(P) = 1$.
* multiplying both sides of $A\underline{x} = \underline{b}$ by P does not change the solution. $PA\underline{x} = P\underline{b}$

Gauss Elimination with Partial Pivoting

We permute the matrix <u>before</u> eliminating below a diagonal ~~so that~~ to choose the lagest possible <u>pivot</u>.

Instead of performs:

$$M_{n-1} \cdots M_2 M_1 A = U$$

we perform.

$$\underbrace{M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A = U}_{*}$$

where P_i are either permutations or the identity

It turns out that $*$ is equivalent to

$$\underbrace{(\hat{M}_{n-1} \hat{M}_{n-2} \cdots \hat{M}_2 \hat{M}_1)}_{\substack{\text{lower triangular.} \\ L^{-1}}} \underbrace{(P_{n-1} \cdots P_1)}_{P.} A = U. \Rightarrow PA = LU$$

product of permutation matrixes is still a permutation matrix

Summary to solve $A\underline{x} = \underline{b}$

① find P, L, U s.t. $PA = LU$.

Gauss Elimination with partial pivoting

② $A\underline{x} = \underline{b} \Leftrightarrow PA\underline{x} = P\underline{b}$

$\Leftrightarrow LU\underline{x} = P\underline{b}$

solve $\quad L\underline{y} = P\underline{b}$

$\qquad\quad U\underline{x} = \underline{y}$

Example $\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$. with $\epsilon < \epsilon_{mach}$.

without pivoting, we get.

$$L = \begin{bmatrix} 1 & 0 \\ 1/\epsilon & 1 \end{bmatrix} \quad U = \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix}$$

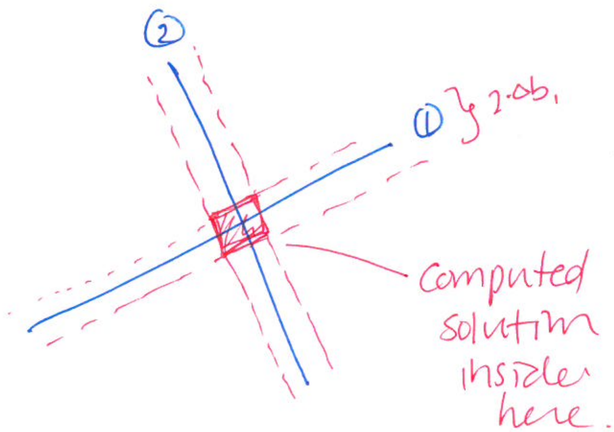
Then $L \cdot U = \begin{bmatrix} 1 & 0 \\ 1/\epsilon & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix} = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix} \neq A$

Geometric Interpretation of conditioning

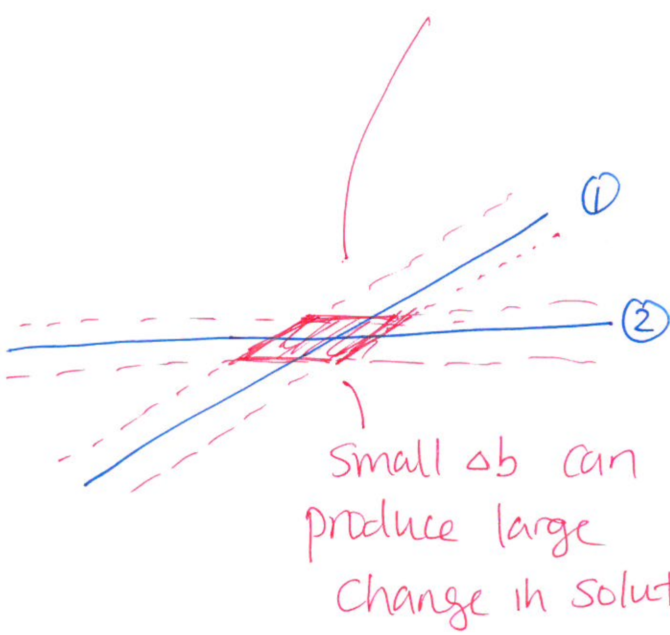
in 2D. $\underline{A}\underline{x} = \underline{b}$ represents finding the intersection of 2 lines.

① $a_{11} x_1 + a_{12} x_2 = b_1$
② $a_{12} x_1 + a_{22} x_2 = b_2$
} lines in 2D.

$a_{11} x_1 + a_{12} x_2 = \underline{b_1 + \Delta b_1}$ lines w/ the slope but slightly different intercepts.



well conditioned.



ill conditioned.