Announcements

If your assignment 1 code didn't run, markus will show a mark of 0.0%. (see Quercus announcements)

You can resubmit your work by tommorrow 9pm (Jan 24) under the markus assignment a1_resubmit with 20% penalty

I will reduce the penalty to 5% if you come to O.H. Jan 25 / 28 / 30 to ask for help or show that you started on A2.

11-12 11-12 1-2 pm.

We might not allow resubmissions on future assignments, so please test your code!

System of Linear Equations:

Algorithms to solve systems of form:

$$\underset{\substack{n\times n\\ \text{matrix}}}{A}\ \underset{\substack{n\times 1\\ \text{vector}}}{\underline{x}} = \underset{\substack{n\times 1\\ \text{vector}}}{\underline{b}}$$

~~for x~~ we are solving for $\underline{x}$ given A and $\underline{b}$

we will focus on the case ~~of~~ $A \in \mathbb{R}^{n\times n}$ square and non-singular.

If $A \in \mathbb{R}^{m\times n}$ rectangular

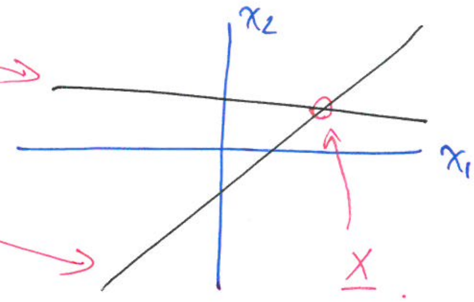
and $m > n$ — we have an overdetermined system

$m < n$ — " " underdetermined system

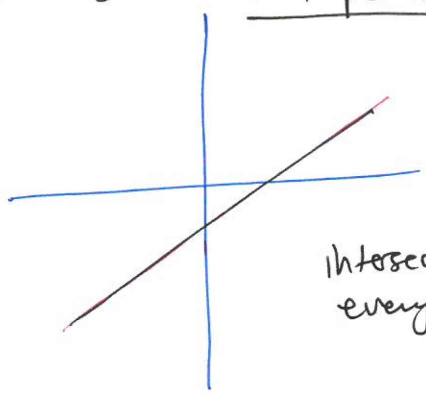
If $A \in \mathbb{R}^{2\times 2}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

then $\underline{x}$ represents the intersection of 2 lines

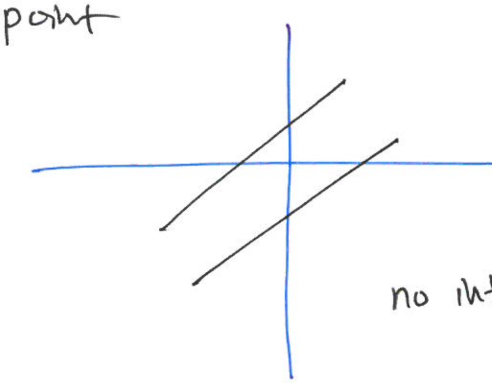
$$A\underline{x} = \underline{b} = \begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$



2 Lines intersect at a unique point if the 2 lines are not parallel:



intersect everywhere



no intersection

Def: A matrix "A" is nonsingular iff there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{bmatrix}$ (i.e. A is invertible)

Equivalents: $\quad \det(A) \neq 0 \qquad A \in \mathbb{R}^{n\times n}$

$\operatorname{rank}(A) = n$

for a vector $\underline{z} \in \underline{0} \qquad A\underline{z} \neq \underline{0}$

If A is nonsingular, then $A\underline{x} = \underline{b}$ has a unique solution.

Examples of singular matrices:

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}}_{\text{Infinite solutions}} \qquad \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}}_{\text{no solution}}$

How do we solve $A\underline{x} = \underline{b}$, $A \in \mathbb{R}^{n \times n}$ nonsingular?

⇒ Why not compute A^{-1}, then set $\underline{x} := A^{-1}\underline{b}$

How do we compute A^{-1}?

Need to solve $\underbrace{AY = I}$ for Y. then $\underline{x} := Y \cdot \underline{b}$

Need to solve $\underbrace{A y_i = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{n\times,\ \text{once for each } i \in \{1,2,\dots,n\}}$ ← i^{th} row.

When is $A\underline{x} = \underline{b}$ easy to solve?

⇒ A is diagonal. $a_{ij} = 0$ if $i \neq j$

eg// $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\underline{x} = \begin{bmatrix} 1/3 \\ 1/2 \\ 3/5 \end{bmatrix}$

Each equation looks like $a_{ii} x_i = b_i$

$$\Rightarrow \boxed{x_i = \frac{b_i}{a_{ii}}}$$

Requires n flops ← floating point operation
to solve

⇒ A is nonsingular lower triangular.

$$\begin{cases} a_{ij} = 0 & \text{if } j > i \\ a_{ii} \neq 0. \end{cases}$$

eg// $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 2 & 2 & 5 \end{bmatrix}$. $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Forward Substitution:

$$x_1 = b_1 / a_{11}$$
$$x_2 = (b_2 - x_1 a_{21}) / a_{22}$$
$$x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j \right) / a_{ii}$$

$x_1 = 1/3$

$2(\underbrace{x_1}_{1/3}) + 4(x_2) = 2$

$x_2 = \dfrac{2 - 2(\frac{1}{3})}{4} = 1/3$

$2x_1 + 2x_2 + 5x_3 = 3$

$x_3 = \dfrac{3 - 2(2) - 2(\frac{1}{3})}{5}$

$\Rightarrow A$ is nonsingular upper triangular

eg// $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

Backward Substitution

$$x_i = \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right) / a_{ii}$$

Idea: transform a matrix A into an upper triangular system with the same solution.

Gauss Elimination

Change the system $A\underline{x} = \underline{b}$ to a system $A'\underline{x} = \underline{b'}$ with the same solution $\underline{x}$

upper triangular.

Example:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \underbrace{\begin{bmatrix} 2 & 1 & \boxed{1} & 0 \\ 4 & 3 & \boxed{3} & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}}_{b}$$

Step 1: Use elementary row operations to zero out these values.

$3 - 2 \cdot 1$

$$\begin{array}{l} \\ R_2 \leftarrow R_2 - 2 \cdot R_1 \\ R_3 \leftarrow R_3 - \frac{8}{2} R_1 \\ R_4 \leftarrow R_4 - \frac{6}{2} R_1 \end{array} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & \boxed{1} & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}}_{A'} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -5 \\ -6 \end{bmatrix}}_{b'}$$

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} A.$$

$b' = M_1 b$

M_1 represents the operations above

call this matrix M_1

Step 2

$$\begin{array}{r} R_3 \leftarrow R_3 - \frac{3}{1} R_2 \\ R_4 \leftarrow R_4 - \frac{4}{1} R_2 \end{array} \quad \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & \boxed{0} & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} \qquad m_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

Step 3

$$R_4 \leftarrow R_4 - \frac{2}{2} R_3 \quad \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix} \qquad m_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Back solve to get $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

One way to interpret what we did was we left multiplied the system $A\underline{x} = \underline{b}$ by m_1, m_2, m_3.

$$(m_3 (m_2 (m_1 A))) \underline{x} = m_3 m_2 m_1 \underline{b}.$$

$$\underbrace{(m_3 m_2 m_1 A)}_{\text{this became upper triangular}} \underline{x} = m_3 m_2 m_1 b$$

The matrizes m_1, m_2, m_3 are called <u>Elementary Elimination Matrizes</u>

Gauss Elimination Algorithm:

Loop over columns k: $1 \ldots (n-1)$
  # Eliminate Below A_{ii}.
  Loop over rows. i: $(k+1 \ldots n)$.

$$m = \frac{a_{ik}}{a_{kk}} \Big\} \text{ the multipler.}$$

$R_i \leftarrow R_i - m \cdot R_k$.

  # $R_i \leftarrow R_i - mR_k$.

  Loop over j: $i+1 \ldots n$:

$$a_{ij} = a_{ij} - m \cdot a_{kj} \qquad O(n^3)$$

Elementary Elimination Matrix

Properties:

1. M_k. lower triangular, have unit diagonal, nonsingular

2. $M_k = I - \underline{m} \cdot \underline{e_k}^T$ $\qquad m = [0, \ldots, 0, m_{k+1}, \ldots, m_n]^T$

$e_k = [0 \ldots 1, 0, \ldots 0]^T$

k^{th} column of the identity matrix $I_{n \times n}$

$$= \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ 0 & & -m_{k+1} & \ddots & \\ & & \vdots & & \\ & & -m_n & & 1 \end{bmatrix}$$

3. $M_k^{-1} = I + me_k^T$

4. $M_j = I - t \cdot e_j^T$

$$M_k M_j = I - me_k^T - te_j^T + m \underbrace{e_k^T t}_{=0} e_j^T$$

$$= I - me_k^T - te_j^T \Rightarrow \text{products are "unions"}$$

$\hookrightarrow$ lower triangular

$$\underbrace{M_3 M_2 M_1}_{\text{lower triangular.}} A = \underbrace{U}_{\text{upper triangular}}$$

$$A = \underbrace{\underbrace{(M_3 M_2 M_1)^{-1}}_{\text{lower triangular}}}_{L} \; \underbrace{\underbrace{U}_{\text{upper triangular.}}}_{U}$$

$$A = LU$$

If we kept track of the elementary elimination matrices while running Gauss Elimination, we can <u>factor</u> A into an upper triangular and a lower triangular component.

And if we have a factorization $A = LU$. then solving $\underline{A}\,\underline{x} = \underline{b}$ is relatively straightforward.

$$LU\underline{x} = \underline{b}:$$

first solve $L\underline{y} = \underline{b}$ using forward substitution

then solve $U\underline{x} = \underline{y}$ using backward substitution.

LU factorization is useful when we want to solve $A\underline{x} = \underline{b}$ for multiple values of $\underline{b}$.