CSC 338 Lecture 8 Non-linear Equations (§5).

Problem: Find x such that $f(x) = y$
where f is any smooth function.
can take any derivative
(for now $f: \mathbb{R} \to \mathbb{R}$, extend to $f: \mathbb{R}^n \to \mathbb{R}^m$)

Equivalently, find root(s) of a smooth function $f(x)$.
i.e. find an x^* such that $f(x^*) = 0$.

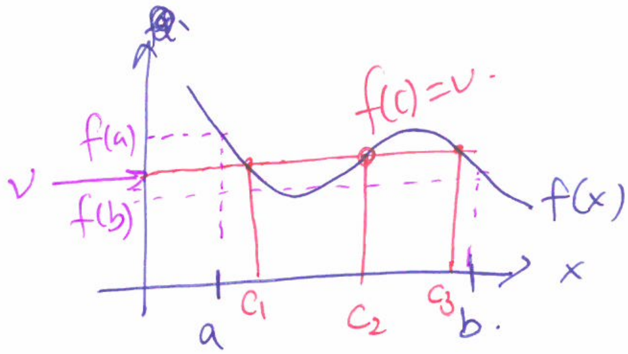
Existence and Uniqueness unlike in a linear system,

$f(x) = 0$ can have
- no solution — $f(x) = x^2 + 1$
- one solution — $f(x) = x$
- many solutions — $f(x) = x^2 - 1$
- infinitely many solutions : $f(x) = \sin(x)$

Even though we can't make global assertions about existence / uniqueness of roots, there are useful local criteria that guarantees the existence of solutions in a region.

Intermediate Value Theorem (IVT)

If $f: \mathbb{R} \to \mathbb{R}$ is continuous on an interval $[a, b]$ and v is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ with $f(c) = v$.

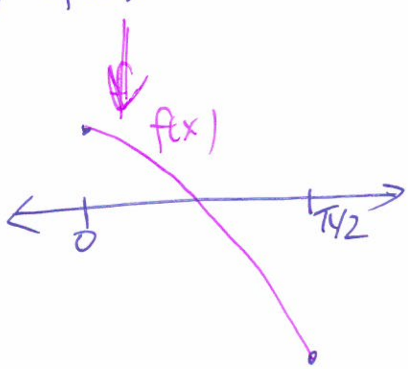


we can apply IVT to root finding:
If f is continuous, f(L) and f(R) have opposite signs, there exists x^* between L, R with $f(x^*) = 0$.

we can use this idea to develop an algorithm for root finding.

eg// $f(x) = \cos(x) - x$. $f(0) = 1$, $f(\frac{\pi}{2}) = -\frac{\pi}{2}$

$\Rightarrow$ there must be a root in the interval $[0, \frac{\pi}{2}]$.



Take midpoint $m_1 = 0 + \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4}$.

$f(m_1) = -0.07829 < 0$

Reapply IVT $\Rightarrow$ root in interval $[0, \frac{\pi}{4}]$

Next midpoint $m_2 = 0 + \frac{\frac{\pi}{4} - 0}{2} = \frac{\pi}{8}$

$f(m_2) = 0.53118 > 0$

$\Rightarrow$ root in interval $[\frac{\pi}{8}, \frac{\pi}{4}]$

Interval Bisection Algorithm

Start with $f: \mathbb{R} \to \mathbb{R}$ continuous and $a, b \in \mathbb{R}$ with $\text{sign}(f(a)) \neq \text{sign}(f(b))$.

Iteratively find $m = a + \frac{(b-a)}{2}$ and if

$\text{sign}(f(m)) = \text{sign}(f(a)) \Rightarrow a := m.$

otherwise $\Rightarrow b := m.$

Aside: midpoint computation.

Why do we compute $m = a + \frac{(b-a)}{2}$ rather than $m = \frac{a+b}{2}$.

Consider $F(\beta=10, p=2, U=10, L=-10)$

$a = 0.67 \longrightarrow 6.7 \times 10^{-1}$

$b = 0.69 \longrightarrow 6.9 \times 10^{-1}$

If we compute $\frac{a+b}{2}$, first compute $a+b$.

$(a+b) = 1.36 \times 10^0 \underset{\text{rounded}}{\Longrightarrow} 1.4 \times 10^0$

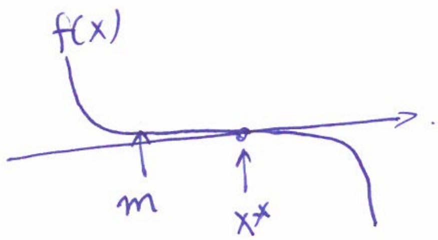
So $\frac{(a+b)}{2} = 0.7 = 7.0 \times 10^{-1}$ in floating-pt arithmetic.

Problem: m is not even between $[a, b]$.

The computation $m = a + \frac{b-a}{2}$ guaranteed to be between $[a, b]$.

Stopping Criteria ⇒ Depend on the problem

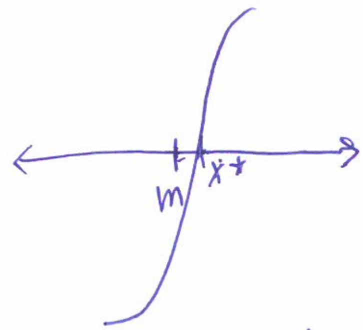
Idea 1 when $f(m)$ is small enough.



$f(m)$ is small

$|m - x^*|$ is large.

Idea 2 when the interval size is small enough.

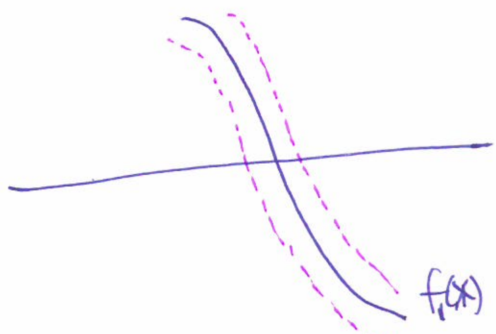


$|m - x^*|$ is small

$f(m)$ is large.

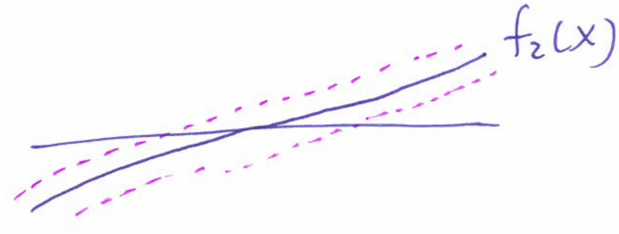
Sensitivity and Conditioning

For which function is the root-finding problem $f(x)=0$ well-conditioned?



small perturbation in f ⇒ small change in solution (root)

well-conditioned



small perturbation in f ⇒ large change in solution (root)

ill-conditioned.

Sensitivity of root finding problem is the <u>opposite</u> to that of evaluating the function $f(x)$

Essentially evaluating $f^{-1}(0)$.

To measure sensitivity, we need to use absolute Condition Numbers because $f(x^*) = 0$

Absolute C.N. for evaluating $y = f(x)$

If Δx change in x then $\Delta y = f(x+\Delta x) - f(x)$

and the ratio

$$\frac{|\text{Abs Error in Sol'n}|}{|\text{Abs Error in Problem}|} = \left|\frac{\Delta y}{\Delta x}\right| = \left|\frac{f(x+\Delta x)-f(x)}{\Delta x}\right| \approx |f'(x)|.$$

Absolute C.N. for root-finding is the inverse: $\dfrac{1}{|f'(x)|}$

If $f'(x^*) = 0$, then x^* is a multiple root

A slight perturbation of f can cause x^* to be one root or none at all!

On the flip side, if C.N. is small then small residual $|f(x)| \Rightarrow |x - x^*|$ is small.

Convergence Rate

Def A series $\{e_1, e_2, \dots\}$ converges to zero if

$$\lim_{k\to\infty} e_k = 0$$

Def A series $\{e_1, e_2, \dots\}$ converges with rate r if

if $\displaystyle\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^r} = C$ for some constant $C > 0$.

Intuition: if k is large enough, $|e_{k+1}| \approx c \cdot |e_k|^r$

In particular, if $r=1$, $c<1$ conv. rate is linear.

$r>1$: : : superlinear.

$r=2$: : : quadratic

$r=3$: : : cubic

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eg//

1. $e_1=10^{-2}$, $e_2=10^{-3}$, $e_3=10^{-4}$, $e_4=10^{-5}$ .... linear, $c=10^{-1}$
2. 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8} ... linear, $c=10^{-2}$
3. 10^{-2} 10^{-3} 10^{-5} 10^{-8} ... superlinear
4. 10^{-2} 10^{-4} 10^{-8} 10^{-16} ... quadratic.

Def. The convergence rate of a root-finding algo is the convergence rate of $e_k = x_k - x^*$

Approximate root at iteration k. True root

An iterative method converges with rate r if $\{e_k\}$ converges with rate r.

For interval bisection, we take the upper bound on $|x_k - x^*|$, i.e. $|e_k| \le \dfrac{|b-a|}{2^k}$

$$\lim_{k\to\infty} \frac{|b-a|/2^{k+1}}{\left(|b-a|/2^k\right)} = \frac{1}{2}.$$

$\Rightarrow$ I.B. converges linearly with $c=\frac{1}{2}$

$\Rightarrow$ One additional bit in accuracy per iteration.

Bisection method converges slowly, but is certain to converge. The method only uses the <u>sign</u> of the function, not its value.

Fixed-Point Iteration

Idea: find an equivalent problem to root finding $f(x)=0$ with the same solution.

<u>Def</u> A point p^* is a <u>fixed-point</u> of $g(x)$ if $g(p^*)=p^*$.

eg// The root of $f(x) = x - 0.2\sin(x) - 0.5$ is the fixed-point of $g(x) = 0.2\sin(x) + 0.5$

If x^* is a root, then

$$x^* - 0.2\sin(x^*) - 0.5 = 0$$

$$\Rightarrow x^* = 0.2\sin(x^*) + 0.5 = g(x^*)$$

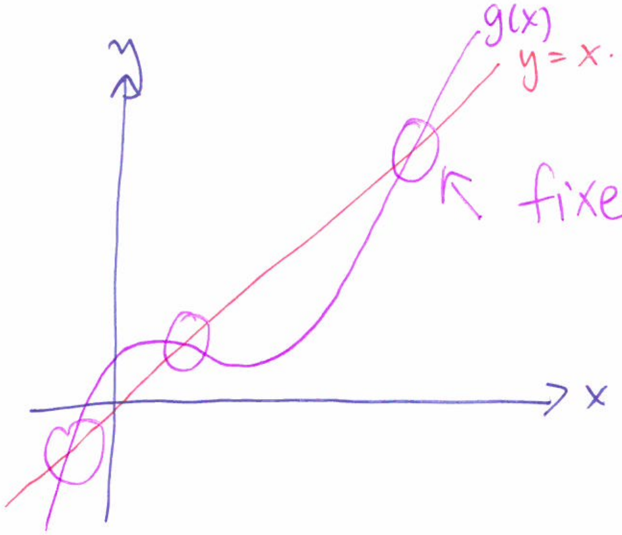
We can sometimes turn a root finding problem into a problem of finding a fixed-point of a related function.

We can find a fixed-point of $g(x)$ using <u>fixed-point iteration</u>

Start with x_0

$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$\vdots$$
$$x_{k+1} = g(x_k).$$
$$\vdots$$

If this iteration converges, then it must converge to a fixed-point of g.



↖ fixed points $y = g(x) = x$.

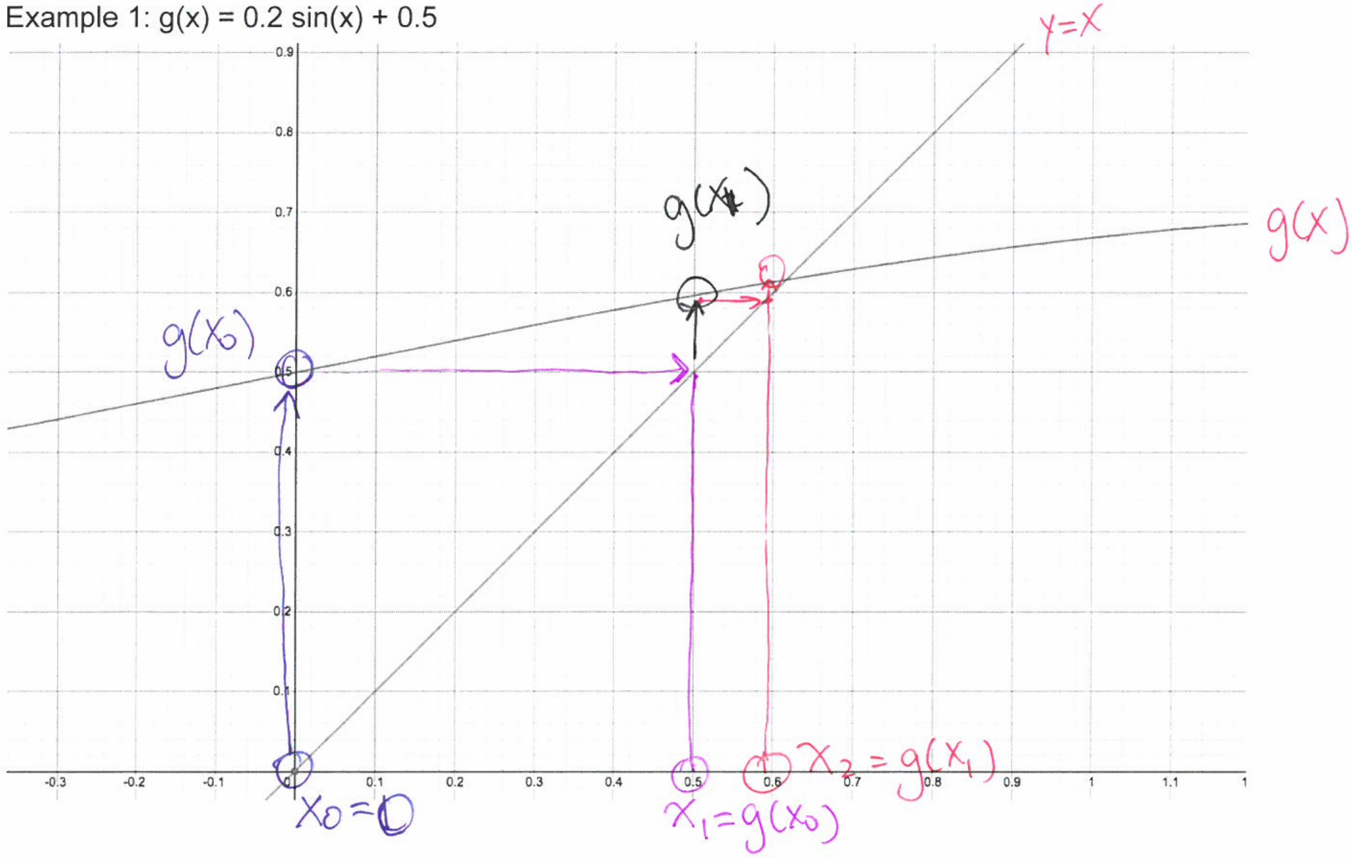
Explanation of handout..

eg// find root of $f(x) = x^3 - x - 1$.

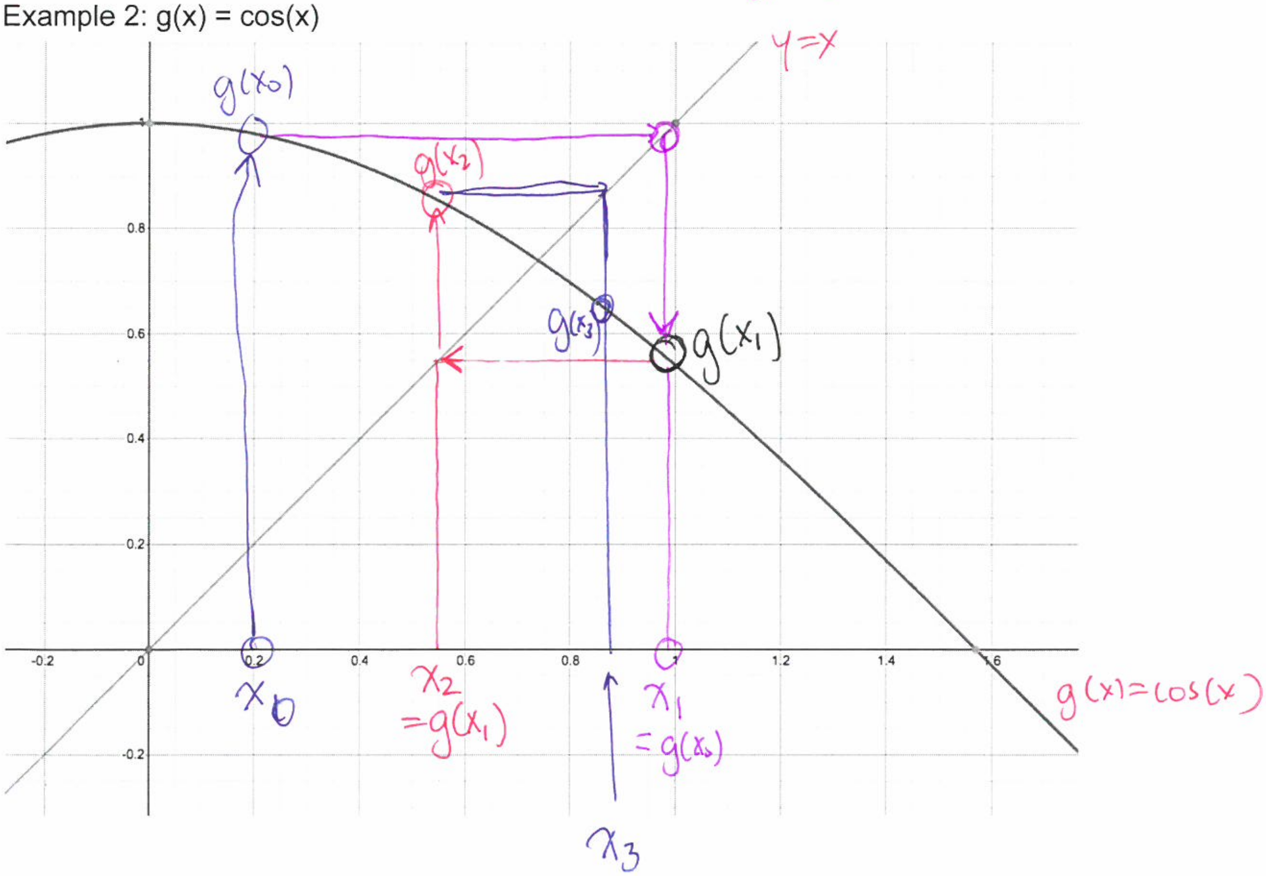
by finding the fixed point of $g(x) = x^3 - 1$

CSC338 Lecture 8 - Fixed-Point Iterations Handout

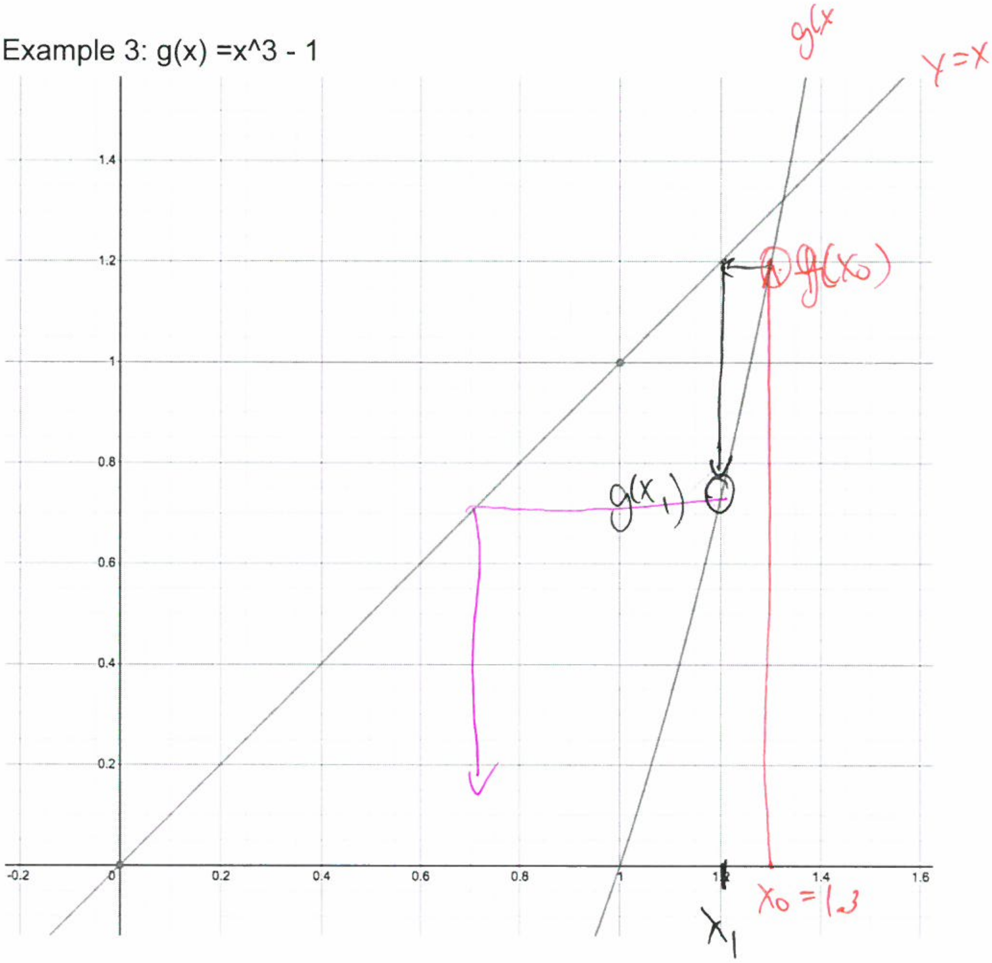
Example 1: g(x) = 0.2 sin(x) + 0.5



Example 2: g(x) = cos(x)



Example 3: g(x) =x^3 - 1



Example 4: g(x) = -x^3 + 5