CSC 338 Lecture 7 (Short Lecture)

We want to solve linear least squares problems:

$$A\underline{x} \approx \underline{b} \qquad A \in \mathbb{R}^{m \times n}, \quad m > n.$$

⇒ Normal Equation (worse conditioning compared to original problem)

⇒ QR Factorization using Householder transformation

↓ ↓

Factor $A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$, $b = Q \begin{bmatrix} \underline{c_1} \\ \underline{c_2} \end{bmatrix}$ (A: $m \times n$, Q: $m \times m$, R: $n \times n$, 0: $(m-n) \times n$)

a.k.a. Elementary Reflectors

where Q orthogonal

R upper triangular

$$H = I - 2\frac{\underline{v}\underline{v}^T}{\underline{v}^T\underline{v}}$$

So $\min_{\underline{x}} \|A\underline{x} - \underline{b}\|_2 = \min_{\underline{x}} \left\| \begin{bmatrix} R \\ 0 \end{bmatrix}\underline{x} - \begin{bmatrix} \underline{c_1} \\ \underline{c_2} \end{bmatrix} \right\|_2$

$= \|\underline{c_2}\|_2$

choose $\underline{v}$ to annhilate elements below the diagonal

eg// $A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$ we will reduce A to the form $A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$

First Step ~ eliminate below the first entry in $\underline{a_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ using a Householder transform.

$$\underline{v_1} = \underline{a_1} \pm \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \leftarrow \|\underline{a_1}\|_2 = \sqrt{1^2 + 2^2 + 2^2} = 3$$

we choose + to avoid cancelation, so

$$\underline{v}_1 = \underline{a}_1 + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

① Verify that $H_1 \underline{a}_1$ eliminates all but the first element $\quad H_1 = I - 2\dfrac{\underline{v}_1 \underline{v}_1^T}{\underline{v}_1^T \underline{v}_1}$

$$H_1 \underline{a}_1 = \underline{a}_1 - 2\,\underline{v}_1 \frac{\underline{v}_1^T \underline{a}_1}{\underline{v}_1^T \underline{v}_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 2 \cdot \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \frac{\cancel{8}\,12}{24} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

② $$H_1 \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \frac{-16+6+4}{24} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

So $$H_1 A = \begin{bmatrix} -3 & -2 \\ 0 & \boxed{4} \\ 0 & \boxed{3} \end{bmatrix}.$$

we don't compute H_1 explicitly.

(But for illustration purposes

$$H_1 = I - 2\frac{\underline{v}_1 \underline{v}_1^T}{\underline{v}_1^T \underline{v}_1} = \begin{bmatrix} -\frac{1}{3} & -2/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix})$$

Second Step ~ eliminate below the first entry in the remaining submatrix. $\quad \underline{a}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \|\underline{a}_2\| = 5$

Take $$\underline{v}_2 = \underline{a}_2 + \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

This choice of $\underline{v}_2$ will map $\underline{a}_2 \longrightarrow \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

we don't need to represent H_2 explicitly,
but.

$$H_2 = \left[\begin{array}{c|c} I & \\ \hline & I - 2\frac{\underline{v}\underline{v}^T}{\underline{v}^T\underline{v}} \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & -3/5 \\ 0 & -3/5 & 4/5 \end{bmatrix}$$

So $H_2 H_1 A = \begin{bmatrix} -3 & -2 \\ 0 & -5 \\ 0 & 0 \end{bmatrix}$. So set $R = \begin{bmatrix} -3 & -2 \\ 0 & -5 \end{bmatrix}$

$$Q = H_1^{-1} H_2^{-1}$$

$$\Rightarrow \quad A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$$