CSC338 Midterm Solutions

Notes

- Some of the grading was done on paper, and others were done on Markus
- The annotations on Markus supersedes the paper grading (we went back to try to give more part marks)

Question 1

1. **False**: We need a *stable* algorithm. The *conditioning* is a property of the problem, not the algorithm
2. **False**: The *truncation* error is the error due to truncating finite series, or taking a fininte series of steps in an infinite process. The truncation error is unchanged if you change a different floating-point system. The *rounding* error is increased.
3. **True**: The under flow level β^L is much smaller than the machine precision β^{1-p} or $\frac{1}{2}\beta^{1-p}$.
4. **True**: If x and y are floating-point numbers, then $x * y$ and $y * x$ are the same.
5. **False**: There was a counter example in homework 3 (or 4).
6. **True**: We have one solution if A is invertible, and infinite solutions otherwise.
7. **False**: The matrix $A = \begin{bmatrix} 0.00001 & 1 \\ 1 & 0.00001 \end{bmatrix}$ is well conditioned but has a small determinant.
8. **True**: This is from lecture 6, page 4 in the notes.

Question 2

Part (a): Continuity condition is necessary because the problem is usually represented using floating-point numbers, so the problem representation will not be exact. Only if the problem is continuous will our computed solution be close to the exact solution.

Part (b): Yes, by dividing a number with an exponent close to U by a number with an exponent close to L (or just something negative).

Part (c):

$$\begin{aligned} A^+ &= (A^T A)^{-1} A^T \\ &= A^{-1} (A^T)^{-1} A^T \\ &= A^{-1} \end{aligned}$$

Question 3

Part (a): The values of the mantissa is 2.23 and the exponent is 1.

$$\begin{aligned} 12.6 &= 2 \cdot 5 + 2 \cdot 5^0 + 3 \cdot 5^{-1} \\ &= (2 + 2\frac{1}{5} + 3\frac{1}{5^2}) \times 5^1 \end{aligned}$$

Some people thought that 1.26 was already in base $\beta = 5$. You can tell this is incorrect for two reasons: (1) the question statement tells us that the number 12.6 is indecimal, and (2) we can't have a digit 6 in base $\beta = 5$.

Part (b):

$$
\begin{array}{lr}
04.31 & \times 5^3 \\
+00.244 & \times 5^3 \\
=10.104 & \times 5^3 \\
=1.01 & \times 5^4
\end{array}
$$

Some people included digits outside the base $\beta = 5$ digits $\{0, 1, 2, 3, 4\}$, or forgot to carry. Some people forgot to align the mantissas of the two inputs based on the difference in the exponents.

Part (c):

We have $\epsilon_{\text{mach}} = \beta^{1-3} = 0.04$, and the relative error $\frac{1.01-1.0104}{1.0104} = -0.00396$.

Question 4

First, apply the permutation matrix to swap rows 3 and 4.

$$
P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P_3 A'' = \begin{bmatrix} 1 & 2 & 5 & 0 & 7 \\ 0 & 2 & 0 & 1 & 3 \\ 0 & 0 & -4 & 1 & 2 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 & 3 \end{bmatrix}
$$

Now the matrix M_3 is below, and when applied the new matrix becomes:

$$
M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} A''' = \begin{bmatrix} 1 & 2 & 5 & 0 & 7 \\ 0 & 2 & 0 & 1 & 3 \\ 0 & 0 & -4 & 1 & 2 \\ 0 & 0 & 0 & 0.75 & 2.5 \\ 0 & 0 & 0 & 1.5 & 5 \end{bmatrix}
$$

Question 5

Part (a)

- First, compute the LU factorization of $PA = LU$
- Then, for each column $\mathbf{b}_k$ of B, compute $x_k = A^{-1}\mathbf{b}_k$ using forward and backward substitution
- The solution matrix has the columns x_k from the previous step

Part (b) See lecture 5 notes.

$$
L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
$$

Question 6

The math in this question is identical to those in lecture 4, page

Part (a): First, $\Delta\mathbf{y} = A(\mathbf{x} + \Delta\mathbf{x}) - A\mathbf{x} = A\Delta\mathbf{x}$

So, using the matrix norm property that $||A\Delta\mathbf{x}|| \leq ||A||||\Delta\mathbf{x}||$, we have $||\Delta\mathbf{y}|| = ||A\Delta\mathbf{x}|| \leq ||A||||\Delta\mathbf{x}||$

Part (b): Since $y = Ax$ and A is invertible, we can write $x = A^{-1}y$. Using the matrix norm property that $||A^{-1}y|| \leq ||A^{-1}||||y||$, we have $||\mathbf{x}|| = ||A^{-1}y|| \leq ||A^{-1}||||y||$

Part (c): Since $||\Delta\mathbf{y}|| \leq ||A||||\Delta\mathbf{x}||$ and $||\mathbf{x}|| \leq ||A^{-1}||||\mathbf{y}||$, we have that

$$\begin{aligned}
\frac{||\Delta\mathbf{y}||}{||A^{-1}||||\mathbf{y}||} &\leq \frac{||A||||\Delta\mathbf{x}||}{||\mathbf{x}||} \\
\frac{||\Delta\mathbf{y}||}{||\mathbf{y}||} &\leq ||A||||A^{-1}||\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \\
\frac{||\Delta\mathbf{y}||}{||\mathbf{y}||} &\leq cond(A)\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}
\end{aligned}$$