Last Name: ________________ **First Name:** ________________

Student #: |_|_|_|_|_|_|_|_|_|_| **Signature:** ________________

UNIVERSITY OF TORONTO MISSISSAUGA
APRIL 2019 FINAL EXAMINATION
CSC338H5S
NUMERICAL METHODS
Lisa Zhang
Duration - 2 hours
Aids: double-sided aid sheet, non-programmable calculator

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Please note, once this exam has begun, you **CANNOT** *re-write it.*

You must earn 40% or above on the exam to pass the course; else, your final course mark will be set no higher than 47%.

This final examination consists of 8 questions on 16 pages (including this page). When you receive the signal to start, please make sure that your copy of the examination is complete.

If you need more space for one of your solutions, use the last pages of the exam and indicate clearly the part of your work that should be marked.

MARKING GUIDE

# 1:	______/ 10
# 2:	______/ 16
# 3:	______/ 8
# 4:	______/ 12
# 5:	______/ 14
# 6:	______/ 15
# 7:	______/ 12
# 8:	______/ 13
TOTAL:	______/100

Good Luck!

Question 1. [10 MARKS]

Circle either "True" or "False" for each of the below statements.

1.	True	False	You can improve the conditioning of a problem by choosing a better algorithm to solve it.
2.	True	False	If we use Newton's method to find the minimum of the function $f(x) = x^2 + 3x - 4$, the method will converge in exactly one iteration.
3.	True	False	If a matrix has a very small determinant, then it has a very high condition number.
4.	True	False	The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ is orthogonal.
5.	True	False	The secant method for root-finding typically converges faster than Newton's method.
6.	True	False	Given a system of linear equations, a small residual and a small condition number guarantees an accurate solution.
7.	True	False	The series 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, ... converges quadratically.
8.	True	False	A linear least squares problem always has a solution.
9.	True	False	For an invertible, square matrix A, we have $A^{-1} = A^{+}$.
10.	True	False	Householder Transformation is preferred because it is more numerically stable than Gauss Elimination.

Question 2. [16 MARKS]

Part (a) [2 MARKS]

What is the (relative) condition number of the problem of evaluating the function $f(x) = x^2 + 1$ at $x = 1$?

Part (b) [2 MARKS]

What is the (relative) condition number of the problem of solving for the vector x that satisfies $Ax = b$, where $A = \begin{bmatrix} 3 & 0 \\ 0 & -0.1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? Use the L_1 norm (the 1-norm).

Part (c) [2 MARKS]

What is the condition number of the problem of finding the root of $f(x) = x^3 - 1$?

Part (d) [2 MARKS]

Consider the problems of minimizing $f_1(x) = (x-3)^8$ and minimizing $f_2(x) = (x-2)^{12}$. Which of the two minimization problems has **worse** conditioning? Briefly (with no more than 10 words) explain why.

Part (e) [2 MARKS]

Write down the permutation matrix that would reverse the rows of a 4×4 matrix.

Part (f) [2 MARKS]

Write down a 4×4 elementary elimination matrix that would perform the operation $R_4 \to R_2 + R_4$ and leave the other rows unchanged.

Part (g) [4 MARKS]

What is the normal equation for the least squares problem $Ax \approx b$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$?

Perform all the computations to set up the normal equation, but do not solve it.

Question 3. [8 MARKS]

Consider the following Python code, where A is an $n \times n$ matrix, and v is a vector of size n.

```
import numpy as np

A_inv    = np.linalg.inv(A)
A_inv_sq = np.matmul(A_inv, A_inv)
b        = np.matmul(A_inv_sq, v)
```

Part (a) [2 MARKS]

Write a mathematical expression relating b to A and v.

$b =$

Part (b) [6 MARKS]

Describe an algorithm to compute b without inverting any matrices. You do not have to write Python code. Instead, explain step by step what operations to perform.

Question 4. [12 MARKS]

Part (a) [6 MARKS]

Consider the floating-point system $F(\beta = 10, p = 4, L = -10, U = 10)$. Suppose we wish to compute the average of three numbers x, y, and z. Consider two algorithms:

```
# Algorithm 1:
avg = (x + y + z) / 3

# Algorithm 2:
avg = x/3 + y/3 + z/3
```

Which algorithm is preferable? Circle your choice above.

Provide example values for x, y and z, and show that your chosen algorithm provides the correct answer, whereas the other algorithm provides an incorrect answer or no answer at all.

Part (b) [6 MARKS]

Consider the normalized floating-point system $F(\beta = 2, p = 6, L = -100, U = 100)$, where chopping is used for rounding. We wish to perform the computation below using the floating-point system.

$$f(n) = (2^n - 10) - ((2^n - 5) - 5)$$

Find one positive integer n such that $f(n)$ evaluates to a nonzero floating-point value. Show the computation and the results of $2^n - 10$ and $(2^n - 5) - 5$.

Question 5. [14 MARKS]

Suppose you are using Householder transformations to compute the QR factorization of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 1 \\ 2 & 3 & 9 \\ 2 & 5 & 5 \\ 3 & 1 & 1 \end{bmatrix}$$

Part (a) [2 MARKS]

How many Householder transformations are required?

Part (b) [4 MARKS]

Specify the first Householder transformation by finding the vector v describing the transformation.

Part (c) [6 MARKS]

Apply the first Householder transformation from Part (b) to the matrix A. Draw a box around your final result.

Part (d) [2 MARKS]

What does the first column of A become as a result of applying the second Householder transformation? (Do not compute the second Householder transformation for the entire matrix.)

Question 6. [15 MARKS]

Consider the function $f(x) = x^2 \sin(x)$. We wish to find a root of $f(x)$.

Part (a) [3 MARKS]

Notice that $f(0.1) = 0.0009983$ and $f(-0.2) = -0.007947$, so we can apply the bisection method beginnig with the interval $[-0.2, 0.1]$. If we wish to find an estimate of the true root x^* accurate within 10^{-6} (i.e. $|x_{est} - x^*| \leq 10^{-6}$), what is the minimum number of bisection method steps required?

Part (b) [2 MARKS]

Write out the Newton's Method iterant for $f(x)$.

$x_{k+1} =$

Part (c) [2 MARKS]

Compute x_1 using Newton's Method assuming $x_0 = 0.1$, accurate to at least 4 significant digits.

$x_1 =$

Part (d) [2 MARKS]

Based on the above result, do you think that the **secant method** will converge if we set $x_0 = 0.1$ and $x_1 = 0.11$? Provide a brief explanation, but do not perform any computations.

Part (e) [6 MARKS]

Consider using fixed-point iteration on $g(x) = x^2 \sin(x) + x$ to find a root of $f(x) = x^2 \sin(x)$. Will the iteration converge? If so, what is the convergence rate? Show your work.

Question 7. [12 MARKS]

Consider the function $f(x) = (x-2)^2(1+\sin(x))$. It is unimodal on the interval $[-2, 0]$.

Perform two iterations of Golden Section search, beginning with $a = -2$ and $b = 0$. Recall that $\tau = 0.618$.

Part (a) [6 MARKS]

Perform the first iteration of Golden Section search, beginning with $a = -2$ and $b = 0$.

$$a = -2$$
$$f(a) = 1.451$$
$$b = 0$$
$$f(b) = 4$$

$x_1 =$

$f(x_1) =$

$x_2 =$

$f(x_2) =$

The minimum is not in the interval:

Reduced interval to search:

Part (b) [6 MARKS]

Perform the second iteration of Golden Section search.

$a =$

$f(a) =$

$b =$

$f(b) =$

$x_1 =$

$f(x_1) =$

$x_2 =$

$f(x_2) =$

The minimum is not in the interval:

Reduced interval to search:

Question 8. [13 MARKS]

Consider the function $f(x_1, x_2) = x_1^2 + 2x_2^2 - x_1x_2 - 3x_1 - 9x_2 + 3$

Part (a) [2 MARKS]

Compute the gradient of the function.

Part (b) [2 MARKS]

What is the critical point of this function? You do not need to show your work.

$x_1 =$ $x_2 =$

Part (c) [2 MARKS]

Compute the Hessian of the function at the critical point.

Part (d) [5 MARKS]

Is the Hessian you found in part (c) positive definite, negative definite, or indefinite? Justify your answer by attempting to perform Cholesky factorization on the matrix. Show your work.

Part (e) [2 MARKS]

Characterize the critical point as a maximum, minimum or saddle point. Justify your answer.

[Use the space below for rough work. This page will ***not*** *be marked, unless you clearly indicate the part of your work that you want us to mark.]*

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