This document contains a list of problems that are useful for making sure that you have the right background for CSC321. The problems are designed to be straightforward to people with an in-depth understanding of the required math.

Each problem is built around one topic that is important in this course. For example, when talking about neural networks and logistic regression, we will be discussion the Sigmoid function $\sigma(x) = \frac{1}{1+exp(-x)}$, and it will be important for students to have an intuitive grasp of its shape. Understanding the techniques that are used for solving the problems in Section 1 will make it easy for students to understand our discussion of the Sigmoid function in class. If you are having trouble with a problem in this problem set, it is likely not enough to find out what the solution is. Instead, you should read up about the entire topic.

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1   Properties of Functions

1.1   Elementary functions

Let

$$f(x) = \exp(2x - 5),$$
$$g(x) = \log(1/x),$$
$$h(x) = x^2.$$

Sketch the functions f, h, and g. Label the axes, and compute the coordinates of at least three points on the sketch without using a calculator.

1.2   Argmin/Argmax

Consider the function

$$f(x) = \log\left(1 + \frac{1}{1 + |x - 10|}\right).$$

Where is this function's maximum? Where is this function's minimum? Do not use calculus.

2   Calculus

2.1   Definition of the derivative

Let

$$f(x) = x^2 + 1.$$

Using the definition of the derivative and limit notation, write down the formula for the derivative of f at point a. Simplify the expression as much as possible.

2.2   The chain rule and optimization

Let

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x - \mu)^2}{2\sigma^2}\right).$$

At what value of x is the function $f(x)$ maximized? Show how to obtain the answer using calculus and how to check whether you obtained the minimum or the maximum. There is no restriction on the value of σ. Why is that important for correctly solving this problem?

3   Linear Algebra

3.1   Matrix multiplication

Re-express the following formula more concisely by using matrix multiplication notation

$$bm + dn + am + zn.$$

In the expression that you obtain, each entry in each matrix should be a single variable. An example of the kind of expression that is required is
$$\begin{bmatrix} a & b \\ d & c \end{bmatrix} \begin{bmatrix} w & y \\ y & z \end{bmatrix}.$$

3.2   Matrix Multiplication Order

Suppose that you want to perform the following matrix computation ABx, where A and B are matrices, and x is a vector. Which order should you perform the multiplication? That is, should you perform $(AB)x$ or $A(Bx)$? Why?

3.3   Basis and dimension

Give an example of a 2-dimensional linear subspace of a 3-dimensional space. Characterize the spaces both geometrically and by specifying their basis vectors.

4   Probability

4.1   Standard distributions: Bernoulli

A fair coin is tossed three times. What is the probability that the outcome of all three tosses is the same?

4.2   Standard distributions: Bernoulli and Binomial

Suppose you sample 20 independent variables from Bernoulli distribution with parameter θ. What is the expected sum? What is the probability that the sum is greater than 15?

4.3   Standard distributions: Bernoulli and Binomial, independence

In the previous problem, we assumed that the variables are independent. Give an example where the variables are not independent (both mathematically, and by describing a plausible real-life scenario where that is the case). In the scenario you described, what is the expected sum of the 20 variables? What is the probability that the sum is greater than 15?

4.4   Independence

Give an example of a probability mass function $P(X, Y)$ such that the random variables X and Y are *not* statistically independent.

4.5   Conditional probability: basic idea

Two fair dice are tossed (independently). Given that the sum of of the outcomes is 6, what is the probability that the outcome of the first die is 3? Given that the sum of the outcomes is s, what is the probability that the outcome for the first die is a? What is the probability that the sum is s and the outcome for the first die is a? What is the probability that the sum is s and the outcome for the first die is a and the outcome for the second die is b?

4.6   Conditional probability: the Monty Hall Problem

At the end of the Let's Make a Deal with Monty Hall game show, a contestant needs to choose between three closed doors. There is a car behind one of the doors, and there are goats behind two of the other doors. The contestant chooses a door, and Monty Hall, who knows where the car is, opens one of the doors that the contestant didn't choose, and reveals that there is a goat behind it. The contestant is then offered to switch to another, still-closed, door. Should the contestant switch? Assume that Monty Hall will always open one of the doors to reveal a goat, regardless of which door the contestant chooses.

Suppose the contestant picked door #1, and Monty Hall revealed that there is a goat behind door# 2. Compute the probability $P(d1 = \text{car}, \text{Monty opens Door \# 2})$, and use that to compute $P(d1 = \text{car}|\text{Monty opens Door \# 2})$. Compute the probability $P(d3 = \text{car}, \text{Monty opens Door\# 2})$, and use that to compute $P(d3 = \text{car}|\text{Monty opens Door\# 2})$.

Why is it necessary to note that Monty Hall will *always* open one of the doors to reveal a goat?

4.7   The Law of Total Probability and Bayes' Theorem

The probability that that a truck hits a house at any given second is 1×10^{-10}. The probability of an earthquake happening at any given second is 1×10^{-11}. The probability of a house jumping if there is an earthquake is 0.7, and the probability of a house jumping if a truck hits it is 0.9.

What is the probability of a house jumping in any given second, assuming (implausibly) that trucks hitting the house and earthquakes or independent events?

Given that the house jumped, what is the probability that an earthquake happened?

Given that the house jumped, what is the probability that there was both an eathquake, and a truck hit the house?