# Chapter 3

# Logical Connectives

## Universal quantification and implication again

So far we have considered an implication to be universal quantification in disguise:

Claim 1: If an employee is male, then he makes less than 55,000.

The English indefinite article "an" signals that this means "Every male employee makes less than 55,000," and this closed sentence is either true or false, depending on the domain of employees. Since there is quantification going on, it's natural to wonder what open sentence is being quantified.

Claim 2: If the employee is male, then he makes less than 55,000.

The English definite article "the" often signals and unspecified value, and hence an open sentence. We could transform Claim 2 into Claim 1 by prefixing it with "For every employee, ..."

Claim 2′: For every employee, if the employee is male, then he makes less than 55,000.

This distinction is probably clearer in symbolic notation. Let $E$ mean the set of employees, predicate $L(e)$ mean that employee $e$ makes less than 55,000, and $M(e)$ mean that employee $e$ is male. The Claims 1 and 2′ correspond to [1], whereas Claim 2 (no prime) corresponds to [2]. Since the claim is about male employees, we are tempted to say $\forall m \in M, L(m)$, however we usually take the approach of setting our domain to the largest universe in which the predicates make sense. We don't want to avoid reasoning about non-males. How do you feel about verifying Claim 2 for all six values in $E$, which are true/false?[3]

Do you feel uncomfortable saying that the implications with false antecedents are true? Implications are strange, especially when we consider them to involve causality (which we don't in logic). Consider:

Claim 3: If it rains in Toronto on June 2, 2007, then there are no clouds.

Is Claim 3 true or false? Would your answer change if you could wait a year? What if you waited a year and June 2 was a completely dry day in Toronto, is Claim 3 true or false?[4]

## Vacuous truth

We use the fact that the empty set is a subset of any set. Let $x \in \mathbb{R}$ (the domain is the real numbers). Is the following implication true or false?

Claim 4a: If $x^2 - 2x + 2 = 0$, then $x > x + 5$.

A natural tendency is to process $x > x+5$ and think "that's impossible, so the implication is false." However, there is no real number $x$ such that $x^2 - 2x + 2 = 0$, so the antecedent is false for every real $x$. Whenever the antecedent is false and the consequent is either true or false, the implication as a whole is true. Another way of thinking of this is that the set where the antecedent is true is empty (vacuous), and hence a subset of every set. Such an implication is sometimes called VACUOUSLY TRUE.

In general, if there are no $P$s, we consider $P \Rightarrow Q$ to be true, regardless of whether there are any $Q$s. Another way of thinking of this is that the empty set contains no counterexamples. Use this sort of thinking to evaluate the claims:[5]

CLAIM 4: All employees making over 80,000 are female.

CLAIM 5: All employees making over 80,000 are male.

CLAIM 6: All employees making over 80,000 have supernatural powers.

## EQUIVALENCE

Suppose Al quits the domain $E$. Consider the claim

CLAIM 6A: Every male employee makes between 25,000 and 45,000.

Is Claim 6a true? What is its converse?[6] Is the converse true? Draw a Venn diagram. The two properties describe the same set of employees; they are EQUIVALENT. In everyday language, we might say "An employee is male if and only if the employee makes between 25,000 and 45,000." This can be decomposed into two statements:

> An employee is male if the employee makes between 25,000 and 45,000.
>
> An employee is male only if the employee makes between 25,000 and 45,000.

Here are some other everyday ways of expressing equivalence:

- $P$ iff $Q$ ("iff" being an abbreviation for "if and only if").
- $P$ is necessary and sufficient for $Q$.
- $P \Rightarrow Q$, and conversely.

You may also hear

- $P$ [exactly / precisely] when $Q$

For example, if our domain is $\mathbb{R}$, you might say "$x^2+4x+4 = 0$ precisely when $x = -2$." Equivalence is getting at the "sameness" (so far as our domain goes) of $P$ and $Q$. We may define properties $P$ and $Q$ differently, but the same members of the domain have these properties (they define the same sets). Symbolically we write $P \Leftrightarrow Q$. So now

> An employee is male $\Leftrightarrow$ he makes between 25,000 and 45,000.

Consider the following

> Only female employees make less than 1,000.

This is the (true) converse of the (false):

> If an employee is female, then she makes less than 1,000.

The first statement can be re-written as its own contrapositive, as a statement about male employees:

> If an employee is male, then he does not make less than 1,000.

Summing up

- $P \Rightarrow Q$ tells us about $Ps$. Its converse tells us about non-$P$s.
- The equivalence $P \Leftrightarrow Q$ can be decomposed into $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$, so it tells us about $P$s, $Q$s, non-$P$s, and non-$Q$s.

In everyday langauge you will hear people confuse (sometimes deliberately) an implication with its converse:

> If you are a criminal, then you have something to hide.
>
> Suppose you have something to hide.
>
> Then you are a criminal.

## Existential quantification

Consider another sort of quantification, EXISTENTIAL QUANTIFICATION:

CLAIM 7: There is an employee who makes less than 15,000.

CLAIM 8: An employee makes more than 100,000.

Although the indefinite article "an" is used here, we don't take it to signal universal quantification in Claim 7, due to the phrase "There is." Claim 8 is a bit ambiguous, and would be clearer if re-written as "Some employee..." How do we prove Claim 7 true?[7] Express this in terms of sets.[8] How do you prove or disprove Claim 8?[9] What does this mean in terms of sets?[10]

Existential quantification can turn an open sentence into a closed sentence (statement): "For some employee, the employee makes less than 15,000." In symbols we write $\exists$, which we pronounce "there exists." If $E$ means the set of employees, and $L(e)$ means that $e$ makes less than 15,000, then we can write (with increasing symbolic content):

- $\exists$ employee, the employee makes less than 15,000.
- $\exists$ employee $e$, $e$ makes less than 15,000.
- $\exists e \in E, L(e)$.

In everyday language existential quantification is expressed as:

> There [is / exists] [a / an / some / at least one] ... [such that / for which] ..., or [For] [a / an / some / at least one] ...

Note that the English word "some" is always used INCLUSIVELY here, so "some object is a $P$" is true if every object is a $P$. When is $\exists x, P(x)$ false?[11] The truth values of $\neg\exists x, P(x)$ and $\forall x, \neg P(x)$ are the same. Apply negation again.[12] Saying Claim 8 is false is the same as saying "Every employee does not make more than 100,000." Recall our test of when $\forall x, P(x)$ is false.[13] When there's a counterexample. Existential quantifiers can restrict the domain being considered:

CLAIM 9: Some female employee makes more than 25,000.

CLAIM 10: There exists a male employee making less than 10,000.

For universal quantification we express the restriction with implication (similar to subset inclusion). For existential quantification we express the restriction with AND (symbolically $\wedge$), which is like intersection of sets. In general,

- "Every $P$ is also a $Q$" becomes $\forall x, P(x) \Rightarrow Q(x)$.
  (What's the difference between this and $\forall x, P(x) \wedge Q(x)$?)
- "Some $P$ is also a $Q$" becomes $\exists x, P(x) \wedge Q(x)$.
  (What's the difference between this and $\exists x, P(x) \Rightarrow Q(x)$?)

Now Claim 9 becomes "$\exists$ employee $e$, $e$ is female and $e$ makes more than 25,000." The existence of example Flo makes Claim 9 true. Claim 10 is false, because no male employee makes less than 10,000. In symbols $\forall$ employees $e$, $e$ male $\Rightarrow \neg$($e$ makes less than 10000). The following are equivalent, and you should become comfortable with reasoning why they are:

- $\neg(\exists x, P(x) \wedge Q(x)) \Leftrightarrow \forall x, (P(x) \Rightarrow \neg Q(x))$.
  In words, "No $P$ is a $Q$" is equivalent to "Every $P$ is a non-$Q$."
- $\neg(\forall x, P(x) \Rightarrow Q(x)) \Leftrightarrow \exists x(P(x) \wedge \neg Q(x))$.
  In words, "Not every $P$ is a $Q$" is equivalent to "There is some $P$ that is a non-$Q$."

## Conjunction (And)

We use $\wedge$ ("and") to combine two sentences into a new sentence that claims that both of the original sentences are true. In our employee database:

Claim 11: The employee makes less than 75,000 and more than 25,000.

Claim 11 is true for Al (who makes 60,000), but false for Betty (who makes 500). If we identify the sentences with predicates that test whether objects are members of sets, then the new $\wedge$ predicate tests whether somebody is in both the set of employees who makes less than 75,000 and the set of employees who make more than 25,000 — in other words, in the intersection. Is it a coincidence that $\wedge$ resembles $\cap$ (only more pointy)?

Notice that, symbolically, $P \wedge Q$ is true exactly when both $P$ and $Q$ are true, and false if only one of them is true and the other is false, or if both are false.

We need to be careful with everyday language where the conjunction "and" is used not only to join sentences, but also to "smear" a subject over a compound predicate. In the following sentence the subject "There" is smeared over "pen" and "telephone:"

Claim 12: There is a pen and a telephone.

If we let $O$ be the set of objects, $p(x)$ mean $x$ is a pen, and $t(x)$ mean $x$ is a telephone, then the obvious meaning of Claim 12 is [14] (There is a pen and there is a telephone). But a pedant who has been observing the trend where phones become increasingly smaller and difficult to use might think Claim 12 means [15] (There is a pen-phone).

Here's another example whose ambiguity is all the more striking since it appears in a context (mathematics) where one would expect ambiguity to be sharply restricted.

> The solutions are:
>
> $x < 10$ and $x > 20$
>
> $x > 10$ and $x < 20$

In the first case the author means the union of two sets in the first case, and the intersection in the second. We use $\wedge$ in the second case, and disjunction $\vee$ ("or") in the first case.

## Disjunction (Or)

The disjunction "or" (written symbolically as $\vee$) joins two sentence into one that claims that at least one of the sentences is true. For example,

> The employee is female or makes less than 45,000.

This sentence is true for Flo (she makes 20,000 and is female) and true for Carlos (who makes less than 45,000), but false for Al (he's neither female, nor does he make less than 45,000). If we viewed this "or'ed" sentence as a predicate testing whether somebody belonged to at least one of "the set of employees who are female" or "the set of employees who earn less than 45,000," then it corresponds to the union. As a mnemonic, the symbols $\vee$ and $\cup$ resemble each other. Historically, the symbol $\vee$ comes from the Latin word "vel" meaning or.

We use $\vee$ to include the case where more than one of the properties is true; that is, we use an INCLUSIVE-OR. In everyday English we sometimes say "and/or" to specify the same thing that this course uses "or" for, since the meaning of "or" can vary in English. The sentence "Either we play the game my way, or I'm taking my ball and going home now," doesn't include both possibilities and is an exclusive-or: "one or the other, but not both." An exclusive-or is sometimes added to logical systems (say, inside a computer), but we'll avoid complicating things here.

## Logical arithmetic

If we identify $\wedge$ and $\vee$ with set intersection and union, it's clear that they are ASSOCIATIVE and COMMUTATIVE, so

$$P \wedge Q \Leftrightarrow Q \wedge P \quad \text{and} \quad P \vee Q \Leftrightarrow Q \vee P$$
$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R \quad \text{and} \quad P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

Maybe a bit more surprising is that we have DISTRIBUTIVE LAWS for each operation over the other:

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$
$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

We can also simplify expressions using IDENTITY and IDEMPOTENCY laws:

IDENTITY: $P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q)$.

IDEMPOTENCY: $P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$

## Multiple quantifiers

Many sentences we want to reason about have a mixture of predicates. For example

CLAIM 13: Some female employee makes more than 25,000.

We can make a few definitions, so let $E$ be the set of employees, $\mathbb{Z}$ be the integers, $\mathbf{sm}(e,k)$ be $e$ makes a salary of more than $k$, and $\mathbf{f}(e)$ be $e$ is female. Now I could rewrite:

CLAIM 13′: $\exists e \in E, \mathbf{f}(e) \wedge \mathbf{sm}(e, 25000)$.

It seems a bit inflexible to combine $e$ making a salary, and an inequality comparing that salary to 25000, particularly since we already have a vocabulary of predicates for comparing numbers. We can refine the above expression so that we let $\mathbf{s}(e,k)$ be $e$ makes salary $k$. Now I can rewrite again:

CLAIM 13″: $\exists e \in E, \exists k \in \mathbb{Z},\ \mathbf{f}(e) \wedge \mathbf{s}(e,k) \wedge k > 25000$.

Notice that the following are all equivalent to Claim 13″:

$$\exists k \in \mathbb{Z},\ \exists e \in E, \mathbf{f}(e) \wedge \mathbf{s}(e,k) \wedge k > 25000$$
$$\exists e \in E,\ \mathbf{f}(e) \wedge (\exists k \in \mathbb{Z}, \mathbf{s}(e,k) \wedge k > 25000)$$
$$\exists e \in E,\ \mathbf{f}(e) \wedge (\exists k \in \mathbb{Z},\ \mathbf{s}(e,k) \wedge k > 25000)$$

So $\wedge$ is commutative and associative, and the two existential quantifiers commute.

## Mixed quantifiers

If you mix the order of existential and universal quantifiers, you may change the meaning of a sentence. Consider the table below that shows who respects who:

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| A | $\diamond$ | | | | | |
| B | | $\diamond$ | $\diamond$ | $\diamond$ | $\diamond$ | $\diamond$ |
| C | | $\diamond$ | $\diamond$ | $\diamond$ | $\diamond$ | $\diamond$ |
| D | | $\diamond$ | $\diamond$ | $\diamond$ | $\diamond$ | |
| E | | $\diamond$ | $\diamond$ | $\diamond$ | | |
| F | | $\diamond$ | $\diamond$ | | | |

If we want to discuss this table symbolically, we can denote the domain of people by $P$, and the predicate "$x$ respects $y$" by $r(x,y)$. Consider the following open sentence:

Claim 14: $\exists x \in P,\ r(x,y)$ (that is "y is respected by somebody.")

If we pre-pended the universal quantifier $\forall y \in P$ to Claim 14, would it be true? As usual, check each element of the domain, column-wise, to see that it is.[16] Symbolically,

Claim 15: $\forall y \in P, \exists x \in P, r(x,y)$

or "Everybody has somebody who respects him/her." You can have different $x$'s depending on the $y$, so although every column has a diamond in some row, it need not be the same row for each column. What would the predicate be that claims that some row works for each column, that a row is full of diamonds?[17] Now we have to check whether there is someone who respects everyone:

Claim 16: $\exists x \in P, \forall y \in P,\ r(x,y)$

You will find no such row. The only difference between Claim 15 and Claim 16 is the order of the quantifiers. The convention we follow is to read quantifiers from left to right. The existential quantifier involves making a choice, and the choice may vary according to the quantifiers we have already parsed. As we move right, we have the opportunity to tailor our choice with an existential quantifier (but we aren't obliged to).

## Negation

We've mentioned negation a few times already, and it is a simple concept, but it's worth examining it in detail. The negation of a sentence simply inverts its truth value. The negation of a sentence $P$ is written as $\neg P$, and has the value true if $P$ was false, and has the value false if $P$ was true.

Negation gives us a powerful way to check our determination of whether a statement is true. For example, we can check that

> All employees making over 80,000 are female.

is true by verifying that its negation is false. The negation of Claim 14 is

> Not all employees making over 80,000 are female.

We cannot find any employees making over 80,000 that are not female (in fact, we cannot find any employees making over 80,000 at all!), so this sentence must be false, meaning the original must be true.

## DeMorgan's Laws

These laws can be verified either by a truth table, or by representing the sentences as Venn diagrams and taking the complement.

Sentence $s_1 \wedge s_2$ is false exactly when at least one of $s_1$ or $s_2$ is false. Symbolically:

$$\neg(s_1 \wedge s_2) \Leftrightarrow (\neg s_1 \vee \neg s_2)$$

Sentence $s_1 \vee s_2$ is false exactly when both $s_1$ and $s_2$ are false. Symbolically:

$$\neg(s_1 \vee s_2) \Leftrightarrow (\neg s_1 \wedge \neg s_2)$$

By using the associativity of $\wedge$ and $\vee$, you can extend this to conjunctions and disjunctions of more than two sentences.

## Implication, bi-implication, with $\neg, \vee$, and $\wedge$

If we shade a Venn diagram so that the largest possible portion of it is shaded without contradicting the implication $P \Rightarrow Q$, we gain some insight into how to express implication in terms of negation and union. The region that we can choose object $x$ from so that $P(x) \Rightarrow Q(x)$ is $\neg P \cup Q$ (if we interpret $\neg P$ as the complement of $P$), and this easily translates to $\neg P \vee Q$. This gives us an equivalence:

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

Now use DeMorgan's law to negate the implication:

$$\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow (\neg\neg P \wedge \neg Q) \Leftrightarrow (P \wedge \neg Q)$$

You can use a Venn diagram or some of the laws introduced earlier to show that bi-implication can be written with $\wedge \vee$, and $\neg$:

$$(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$$

DeMorgan's law tells us how to negate this:

$$\neg(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee (\neg P \wedge \neg Q)) \Leftrightarrow \cdots \Leftrightarrow ((\neg P \wedge Q) \vee (P \wedge \neg Q))$$

## Moving negation in

Sometimes things become clearer when negation applies directly to the simplest predicates we are discussing. Consider

Claim 17: $\forall x, \exists y, P(x, y)$

What does it mean for Claim 17 to be false, i.e. $\neg(\forall x, \exists y, P(x, y))$? It means there is some $x$ for which the remainder of the sentence is false:

$$\neg(\forall x, \exists y, P(x, y) \Leftrightarrow \exists x, \neg\exists y, P(x, y)$$

So now what does the negated sub-sentence mean? It means there are no $y$'s for which the remainder of the sentence is true:

$$\exists x, \neg\exists y, P(x, y) \Leftrightarrow \exists x, \forall y, \neg P(x, y)$$

There is some $x$ that for every $y$ makes $P(x, y)$ false. As negation ($\neg$) moves from left to right, it flips universal quantification to existential quantification, and vice versa. Try it on the symmetrical counterpart $\exists x, \forall y, P(x, y)$, and consider

$$\neg(\exists x, \forall y, P(x,y)) \Leftrightarrow \forall x, \neg\forall y, P(x,y)$$

If it's not true that there exists an $x$ such that the remainder of the sentence is true, then for all $x$ the remainder of the sentence is false. Considering the remaining subsentence, if it's not true that for all $y$ the remainder of the subsentence is true, then there is some $y$ for which it is false:

$$\neg(\exists x, \forall y, P(x,y)) \Leftrightarrow \forall x, \exists y, \neg P(x,y)$$

For every $x$ there is some $y$ that makes $P(x,y)$ false.

Try combining this with implication, using the rule we discussed earlier, plus DeMorgan's law:

$$\neg(\exists x, \forall y(P(x,y) \Rightarrow Q(x,y))) \Leftrightarrow \neg(\exists x, \forall y(\neg P(x,y) \vee Q(x,y)))$$

## Proliferating domains

Multiple quantification taxes our picture-drawing skills. For example, we started last lecture with domains $E$ (for employees) and $\mathbb{N}$ (natural number), and we had predicates $f(e)$ (employee is female), $s(e,k)$ (employee makes salary $k$) and $k > 25,000$. If we wanted to guide our thinking with a detailed drawing, we would have to think of our predicates as being in the domain $E \times \mathbb{N}$ — the Cartesian product of $E$ and $\mathbb{N}$, defined as the set of ordered pairs $(e,n)$, where $e \in E$ and $n \in \mathbb{N}$. A trivial change to our predicate can make them artificially take pairs as arguments: $f(e,k)$ means $e$ is female, regardless of $k$, and $(e,k) > 25,000$ means. So now the sentence $\exists e, \exists k, f(e,k) \wedge s(e,k) \wedge (e,k) > 25,000$ simply means that the intersection of the three sets in $E \times \mathbb{N}$ is non-empty.

If we take the Cartesian product of two sets, then we can think of the sets as axes of the two-dimensional plane and make another sort of drawing. From this point of view, the claim $\forall x \in D_1, \exists y \in D_2, P(x,y)$ says that there is a graph of some function in from $D_1$ to $D_2$ that is contained in the set that satisfies $P(x,y)$. On the other hand $\exists y, \forall x, P(x,y)$ says that there is a constant function from $D_1$ to $D_2$ that is contained in the set that satisfies $P(x,y)$. You may need to extend the usual conventions of diagrams to guide your thinking when reasoning about domains.

## Transitivity of implication

Consider $(P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$ (I have put the parentheses to make it explicit that the implications are considered before the $\wedge$). What does this statement imply if considered in terms of sets $P$, $Q$, and $R$?[18] We can also work this out using the logical arithmetic rules we introduced above: write $\neg(((P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))) \Rightarrow (Q(x) \Rightarrow R(x)))$ using only $\vee, \wedge$, and $\neg$, and show that it is a contradiction (never true). Use DeMorgan's law, the distributive laws, and anything else that comes to mind. Thus, implication is transitive.

A similar transformation is that $P(x) \Rightarrow (Q(x) \Rightarrow R(x))$ is equivalent to $(P(x) \wedge Q(x)) \Rightarrow R(x)$. Notice this is stronger than the previous result (an equivalence rather than an implication). We'll prove this statement a little later with the help of truth tables.

## Symbolic grammar

To make our life easier, we impose strict conditions on how we express statements symbolically. A syntactically correct sentence is sometimes called a well-formed formula (abbreviated wff). Note that syntactic correctness has nothing to do with whether a sentence is true or false, or whether a sentence is open or closed. The syntax (or grammar rules) for our symbolic language can be summarized as follows:

- Any predicate is a wff.
- If $P$ is a wff, so is $\neg P$.
- If $P$ and $Q$ are wffs, so is $(P \wedge Q)$.
- If $P$ and $Q$ are wffs, so is $(P \vee Q)$.
- If $P$ and $Q$ are wffs, so is $(P \Rightarrow Q)$.
- If $P$ and $Q$ are wffs, so is $(P \Leftrightarrow Q)$.
- If $P$ is a wff (possibly open in variable $x$) and $D$ is a set, then $\forall x \in D, P$ is a wff.
- If $P$ is a wff (possibly open in variable $x$) and $D$ is a set, then $\exists x \in D, P$ is a wff.
- Nothing else is a wff.

These rules are recursive, and tell us how we're allowed to build arbitrarily complex sentences in our symbolic language. The first rule is called the base case and specifies the most basic sentence allowed. The following rules are recursive or inductive rules: they tell us how to create a new legal sentence from smaller legal sentences. The last rule is a closure rule, and says we've covered everything.

The technical definition of predicate logic is slightly more complicated than what we've described here (we need to talk about bound and free variables, which domains variables belong to, and so on), but that's beyond our scope for the moment.

## Truth tables

A powerful tool for verifying truth of non-quantified statements is the use of TRUTH TABLES. In a truth table, we write all possible truth values for the predicates (how many rows do you need?[19]), and compute the truth value of the statement under each of these truth assignments. Each of the logical connectives yield the following truth tables.

| $P$ | $\neg P$ |
|---|---|
| T | F |
| F | T |

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T | F |
| F | F | F | F | T | T |

We often break complex statements into simpler substatements, compute the truth value of the substatements, and combine the truth values back into the more complex statements. For example, we can verify the equivalence

$$(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$$

using the following truth table:

| $P$ | $Q$ | $R$ | $Q \Rightarrow R$ | $P \Rightarrow (Q \Rightarrow R)$ | $P \wedge Q$ | $(P \wedge Q) \Rightarrow R$ | $(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | T | T | F | T | T |
| F | T | T | T | T | F | T | T |
| F | T | F | F | T | F | T | T |
| F | F | T | T | T | F | T | T |
| F | F | F | T | T | F | T | T |

Since the rightmost column is always true, our statement is a law of logic, and we can use it when manipulating our symbolic statements.

## Summary of manipulation rules

The following is a summary of the basic laws and rules we use for manipulating formal statements. Try proving each of them using Venn diagrams or truth tables.

| | |
|---|---|
| commutative laws | $P \wedge Q \Longleftrightarrow Q \wedge P$ |
| | $P \vee Q \Longleftrightarrow Q \vee P$ |
| | $(P \Leftrightarrow Q) \Longleftrightarrow (Q \Leftrightarrow P)$ |
| associative laws | $(P \wedge Q) \wedge R \Longleftrightarrow P \wedge (Q \wedge R)$ |
| | $(P \vee Q) \vee R \Longleftrightarrow P \vee (Q \vee R)$ |
| distributive laws | $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$ |
| | $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$ |
| contrapositive | $P \Rightarrow Q \Longleftrightarrow \neg Q \Rightarrow \neg P$ |
| implication | $P \Rightarrow Q \Longleftrightarrow \neg P \vee Q$ |
| equivalence | $(P \Leftrightarrow Q) \Longleftrightarrow (P \Rightarrow Q) \wedge (Q \Rightarrow P)$ |
| double negation | $\neg(\neg P) \Longleftrightarrow P$ |
| DeMorgan's laws | $\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$ |
| | $\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$ |
| implication negation | $\neg(P \Rightarrow Q) \Longleftrightarrow P \wedge \neg Q$ |
| equivalence negation | $\neg(P \Leftrightarrow Q) \Longleftrightarrow \neg(P \Rightarrow Q) \vee \neg(Q \Rightarrow P)$ |
| quantifier negation | $\neg(\forall x \in D, P(x)) \Longleftrightarrow \exists x \in D, \neg P(x)$ |
| | $\neg(\exists x \in D, P(x)) \Longleftrightarrow \forall x \in D, \neg P(x)$ |
| quantifier distributive laws | $\forall x \in D, P(x) \wedge Q(x) \Longleftrightarrow (\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$ |
| | $\exists x \in D, P(x) \vee Q(x) \Longleftrightarrow (\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$ |

# Chapter 3 Notes

[1]$\forall e \in E, M(e) \Rightarrow L(e)$

[2]$M(e) \Rightarrow L(e)$

[3]We need to verify the following claims:

- If Al is male, then Al makes less than 55,000.
- If Betty is male, then Betty makes less than 55,000.
- If Carlos is male, then Carlos makes less than 55,000.
- If Doug is male, then Doug makes less than 55,000.
- If Ellen is male, then Ellen makes less than 55,000.
- If Flo is male, then Flo makes less than 55,000.

[4]True, regardless of the cloud situation. In logic $P \Rightarrow Q$ is false exactly when $P$ is true and $Q$ is false. All other configurations of truth values for $P$ and $Q$ are true (assuming that we can evaluate whether $P$ and $Q$ are true or false).

[5]All these claims are true, although possibly misleading. Any claim about elements of the empty set is true, since there are no counterexamples.

[6]Every employee who makes between 25,000 and 45,000 is male.

[7]Look up the table entry for Betty.

[8]The set of employees making less than 15,000 is not empty.

[9]You have to check every employee. The absence of EXAMPLES (rather than COUNTEREXAMPLES) makes it false.

[10]The set of employees earning over 100,000 is empty.

[11]When $\forall x, \neg P(x)$ is true.

[12]So $\exists x, P(x)$ is equivalent to $\neg \forall x, \neg P(x)$. We could live without existential quantifiers, but would our life be as good?

[13]When $\exists x, \neg P(x)$ is true. Notice the symmetry with the test of when $\forall x, P(x)$ is false, that is when $\exists x, \neg P(x)$.

[14]$\exists x \in O,\ p(x) \wedge \exists y \in O,\ t(y)$, or even $\exists x \in O$, $\exists y \in O$, $p(x) \wedge t(y)$.

[15]$\exists x \in O,\ p(x) \wedge t(x)$

[16]True, theres a diamond in every column.

[17]If we were thinking of the row corresponding to $x$, then $\forall y \in P$, $r(x, y)$.

[18]It implies that $P$ is a subset of $R$, since $P \subseteq Q$ and $Q \subseteq R$. It is not equivalent, since you can certainly have $P \subseteq R$ without $P \subseteq Q$ or $R \subseteq Q$.

[19]If you have $n$ predicates, you need $2^n$ rows (every combination of T and F).