This is a **closed-book test**: no books, no notes, no computers (of any kind), no tablets, no calculators, no phones, etc. allowed. The only aid permitted is an 8.5 by 11 inch **aid sheet**. You may write (as small as you like) on both sides of the aid sheet, but you cannot have any "attachments" to the aid sheet.

Do **NOT** turn this page over until you are **TOLD** to start.

Duration of the test: 50 minutes (1:10 to 2:00 PM).

Write your answers to **ALL** questions in the test booklets provided.

Please fill-in **ALL** the information requested on the front cover of **EACH** test booklet that you use.

The test consists of 3 pages, including this one. Make sure you have all 3 pages.

The test consists of 2 questions. **Answer both questions**. Each question consists of two parts; each part is worth 5 marks.

The test was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity**.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

1. [10 marks: 5 marks for each part]

Consider the quadratic function

$$f(x) = \frac{1}{2}x^T Q x - x^T b + c \tag{1}$$

where $c \in \mathbb{R}$, x and $b \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ is symmetric positive-definite. For given vectors $x_k \in \mathbb{R}^n$ and $p_k \in \mathbb{R}^n$ with $p_k \neq 0$, let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(\alpha) = f(x_k + \alpha p_k)$$

(a) Show that $\phi(\alpha)$ has a unique minimum at

$$\alpha_k = -\frac{p_k^T \nabla f(x_k)}{p_k^T Q p_k} \tag{2}$$

where $\nabla f(x)$ is the gradient of $f(x)$.

Show all your calculations.

(b) The Goldstein condition for *sufficient decrease* is

$$f(x_k) + (1-\gamma)\,\alpha_k\, p_k^T \nabla f(x_k) \leq f(x_k + \alpha_k p_k) \leq f(x_k) + \gamma\, \alpha_k\, p_k^T \nabla f(x_k) \tag{3}$$

for a constant $\gamma \in (0, \frac{1}{2})$. Show that, for $f(x)$ given by (1) with Q symmetric positive-definite and α_k given by (2) with $p_k \neq 0$, the Goldstein condition (3) is satisfied for all $\gamma \in (0, \frac{1}{2})$.

Show all your calculations.

Note that, in part (b), you can use α_k given by (2) even if you didn't prove it in part (a).

2. [10 marks: 5 marks for each part]

Suppose $f : \mathbb{D} \to \mathbb{R}$, where $\mathbb{D}$ is an open and convex subset of $\mathbb{R}^n$. Suppose also that $\nabla f(x)$ exists and is continuous for all $x \in \mathbb{D}$.

Recall that $f(x)$ is convex on $\mathbb{D}$ if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all x and $y \in \mathbb{D}$ and all $\alpha \in [0,1]$.

(a) Show that, if $f(x)$ is convex on $\mathbb{D}$, then

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all x and $y \in \mathbb{D}$.

(b) Show that, if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all x and $y \in \mathbb{D}$, then $f(x)$ is convex on $\mathbb{D}$.

Note that you were told above to assume that $\nabla f(x)$ exists and is continuous for all $x \in \mathbb{D}$. If you cannot prove part (a) or part (b) with this assumption alone, then you might choose to assume in addition that $\nabla^2 f(x)$ exists and is continuous for all $x \in \mathbb{D}$. However, if you make this additional assumption in either part (a) or (b), then there will be a two mark deduction in each part that you use this additional assumption. In particular, if you use this additional assumption in both parts (a) and (b), then there will be a total of a four mark deduction for this question.