Duration — 50 minutes

Do **NOT** turn this page over until you are **TOLD** to start.

Answer **ALL** Questions in the Test Booklets Provided

This is a **closed-book test**: no books, no notes, no calculators, no phones, no tablets, no computers (of any kind) allowed.

Please fill-in **ALL** the information requested on the front cover of **EACH** test booklet that you use.

The test consists of 3 pages, including this one. Make sure you have all 3.

The test consists of 2 questions. **Answer both questions**. The mark for each question is listed at the start of the question. Do the questions that you feel are easiest first.

The test was written with the intention that you would have ample time to complete it. You will be rewarded for concise well-thought-out answers, rather than long rambling ones. **We seek quality rather than quantity**.

Moreover, an answer that contains relevant and correct information as well as irrelevant or incorrect information will be awarded fewer marks than one that contains the same relevant and correct information only.

Write legibly. Unreadable answers are worthless.

You may find the following definitions useful.
(Assume $h \in \mathbb{R}$ is a small positive stepsize.)

The *shift* operator:	$\mathcal{E}z(x) = z(x+h)$
The *forward difference* operator:	$\Delta_+ z(x) = z(x+h) - z(x)$
The *backward difference* operator:	$\Delta_- z(x) = z(x) - z(x-h)$
The *central difference* operator:	$\Delta_0 z(x) = z(x+h/2) - z(x-h/2)$
The *averaging* operator:	$\Upsilon_0 z(x) = \frac{1}{2}\Big(z(x+h/2) + z(x-h/2)\Big)$
The *derivative* operator:	$Dz(x) = z'(x)$

1. [5 marks]

   Show that, for any integer $s \geq 1$, if $z^{(s+3)}(\hat{x})$ exists and is continuous for all $\hat{x}$ in an open interval containing both x and $x-h$, then

   $$\frac{1}{h^s}\left(\Delta_-^s + \frac{s}{2}\,\Delta_-^{s+1}\right) z(x) = z^{(s)}(x) + c_s h^2 z^{(s+2)}(x) + \mathcal{O}(h^3)$$

   where $c_s \in \mathbb{R}$ is a constant that depends on s.

   Also give the value of c_s.

2. [15 marks: 5 marks for each of the parts (a), (b) and (c)]

Consider the two-point boundary value problem

$$\begin{aligned} &-y''(x) + y'(x) + y(x) = f(x), \qquad x \in (0,1) \\ &y(0) = \alpha, \qquad y(1) = \beta \end{aligned} \tag{1}$$

Suppose that we discretize (1) as follows:

$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_i - y_{i-1}}{h} + y_i = f_i \qquad \text{for } i = 1, 2, \ldots, m \tag{2}$$

where m is a positive integer, $h = \frac{1}{m+1}$, $y_0 = y(0) = \alpha$, $y_i \approx y(ih)$, for $i = 1, 2, \ldots, m$, $y_{m+1} = y(1) = \beta$ and $f_i = f(ih)$ for $i = 1, 2, \ldots, m$.

(a) To find the unknowns y_i, $i = 1, 2, \ldots, m$, in (2), we usually form a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ and solve $\mathbf{Ay} = \mathbf{b}$ for the vector $\mathbf{y} = [y_1, y_2, \ldots, y_m]^T$.

   i. What are the values of the elements $\mathbf{A}_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$, of the matrix $\mathbf{A}$?

   ii. What are the values of the elements $\mathbf{b}_i$, $i = 1, 2, \ldots, m$, of the vector $\mathbf{b}$?

(b) What is the order of consistency of the numerical method (2)?
Justify your answer.

(c) Is the matrix $\mathbf{A}$ that you found in part (a) nonsingular?
Justify your answer.