This assignment is due at the start of our lecture on Friday, 15 March 2019.

The main part of this assignment consists of questions 1 to 5, which are out of a total of 45 marks. Question 6 is a bonus question, which is optional. If you do any or all of the bonus question, you can earn back marks that you lost on questions 1 to 5. However, the maximum mark that you can earn on this assignment is 45.

For the questions that require you to write a MatLab program, hand in the program and its output as well as any written answers requested in the question. Your program should conform to the usual CS standards for comments, good programming style, etc. Try to format your output from your program so that it is easy for your TA to read your results.

Before writing your MatLab programs, you might find it useful to read the MatLab documentation on our course webpage `http://www.cs.toronto.edu/~krj/courses/446-2310/`. You should use sparse matrices as much as possible. Read help in MatLab on sparfun, sparse and spdiags. After you have initialized a matrix using the MatLab sparse matrix routines, you can solve a system $Ax = b$ in MatLab by `x = A \ b`. MatLab will use an efficient sparse matrix factorization to solve the system.

Everyone who has registered for this course should have an account on CS Teaching Labs Computer System (i.e., formerly called the CDF System). You should be able to access the system remotely over the internet. There is more information about accessing your account at the start of Assignment 1. If you have any trouble with this, let me know and I will try to help.

Throughout this assignment, I refer to MatLab, but you can use Octave or one of the other MatLab clones instead. (See "MatLab Clones" on the course webpage.) However, MatLab clones are not 100% compatible with MatLab. So, run each of your final programs through MatLab to make sure that your program really runs under MatLab, since, when your TA marks your program, he will want to see a working MatLab program.

1. [5 marks]

   Do question 9.2 on page 202 of your textbook.

2. [10 marks]

   Use our corrected version of the Geršgorin criterion (Lemma 8.3 on page 157 of your textbook) to solve question 9.3 on page 202 of your textbook.

   State any additional assumptions that you need to ensure that the system is nonsingular.

   Recall that we already proved in class that this system is nonsingular by showing that the associated matrix is symmetric positive-definite, hence nonsingular. The point of this question is to use the Geršgorin criterion to give an alternative proof of this result.

3. [5 marks]

   Do question 9.4 on page 202 of your textbook.

   The definition of *positive definite* is given on page 185 of your textbook.

   In this case, if the domain associated with the differential operator $\mathcal{L}$ is $[c, d]$, then a function $v(x)$ satisfies the Dirichlet boundary conditions if

$$v(c) = v'(c) = v(d) = v'(d) = 0$$

   That is, both the function and its first derivative are zero at the two endpoints of the domain of $\mathcal{L}$.

4. [15 marks]

   Do question 9.7 on pages 202 and 203 of your textbook.

   This question proves the remark following equation (9.12) on page 179 of your textbook that $\inf_{v \in \mathbb{H}} \mathcal{J}(v) > -\infty$.

   Note that all functions $v \in \mathbb{H}$ satisfy the boundary conditions $v(0) = \alpha$ and $v(1) = \beta$ as well as

$$\int_0^1 [v(x)]^2 \, dx < \infty \qquad \text{and} \qquad \int_0^1 [v'(x)]^2 \, dx < \infty$$

   Moreover,

$$\tilde{a}(v, v) = \int_0^1 \left\{ a(x)[v'(x)]^2 + b(x)[v(x)]^2 \right\} \, dx$$

   where $a(x) > 0$ and $b(x) \geq 0$ for all $x \in [0, 1]$. In addition,

$$\langle f, v \rangle = \int_0^1 f(x) \, v(x) \, dx$$

   and

$$\|v\| = \sqrt{\langle v, v \rangle}$$

   where

$$\langle v, v \rangle = \int_0^1 [v(x)]^2 \, dx$$

   Moreover, you can assume that $a(x)$, $b(x)$ and $f(x)$ are "well-behaved" functions and that all the integrals that you need in this problem are well-defined and finite.

5. [10 marks]

Consider the two-point boundary value problem

$$\begin{aligned} &-y''(x) + \lambda^2 y(x) = x, \quad \text{for } x \in (0,1), \\ &y(0) = y(1) = 1 \end{aligned} \tag{1}$$

It is easy to verify that the solution to this problem is

$$y(x) = c_1 \mathrm{e}^{\lambda x} + c_2 \mathrm{e}^{-\lambda x} + x/\lambda^2$$

where

$$\begin{aligned} c_2 &= \frac{\mathrm{e}^{\lambda} + 1/\lambda^2 - 1}{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}} \\ c_1 &= 1 - c_2 \end{aligned}$$

In this assignment, use $\lambda = \sqrt{10}$ (i.e., $\lambda^2 = 10$).

Write a MatLab program that uses the Ritz-Galerkin method with piecewise linear chapeau (i.e., hat) basis functions (defined on page 176 of your textbook) on an equally spaced grid to solve the two-point boundary value problem (1) with $\lambda^2 = 10$. That is, let the gridpoints be $x_i = ih$ for $i = 0, 1, \ldots, m+1$ and $h = 1/(m+1)$, where m is an integer. (See below for the choices of m.)

The approximate solution generated by the Ritz-Galerkin method has the form

$$y_m(x) = \varphi_0(x) + \sum_{k=1}^{m} \gamma_k \varphi_k(x)$$

where $\varphi_0(x)$ is any simple function that satisfies the boundary conditions associated with the problem (1) (i.e., $\varphi_0(0) = \varphi_0(1) = 1$), $\varphi_k(x)$ is the chapeau basis function defined on page 176 of your textbook and the γ_k are determined by solving the Galerkin equations (see (9.7) on page 174 of your textbook).

For each of $m = 9, 19, 39, 79, 159, 319, 639$, use your program to compute the Ritz-Galerkin solution $y_m(x)$ to the two-point boundary value problem (1) with $\lambda^2 = 10$.

The maximum error in the numerical solution at the gridpoints $\{x_i : i = 1, \ldots, m\}$,

$$\max \{|y(x_i) - y_m(x_i)| : i = 1, \ldots, m\} \tag{2}$$

is a good approximation to the infinity norm of the error in the numerical solution

$$\|y - y_m\|_\infty = \max \{|y(x) - y_m(x)| : x \in [0,1]\}$$

Compute and print the maximum error in the numerical solution at the gridpoints (2) for $m = 9, 19, 39, 79, 159, 319, 639$.

How does this error decrease with $h = 1/(m+1)$?

(Note that, for $m = 9, 19, 39, 79, 159, 319, 639$, $m+1 = 10, 20, 40, 80, 160, 320, 640$ and $h = 1/(m+1) = 1/10, 1/20, 1/40, 1/80, 1/160, 1/320, 1/640$. So, h is halved for each successive m.)

6. Bonus Question

[10 marks if you solve this problem yourself; 5 marks if you can find a published solution in English[1]]

One CSC 446/2310 student found a very nice solution to Problems 7 and 8 on Assignment 1. I'll email the source to you. However, the solution depends on the following definition and theorem.

(I changed the wording just slightly from the version I'll send you.)

Definition 1 *An $n \times n$ matrix A is said to be an M-matrix, if each off-diagonal element of the matrix A is non-positive ($a_{ij} \leq 0$ for all $i \neq j$) and there exists a positive vector r (i.e., $r_i > 0$ for all $i = 1, 2, \ldots, n$) such that $Ar > 0$ (i.e., if $Ar = s$, then $s_i > 0$ for all $i = 1, 2, \ldots, n$).*

Theorem 2 (This is Theorem 1.2.2 in the source I'll send you.)

If $A \in \mathbb{R}^{n \times n}$ is an M-matrix, then

P1: A is nonsingular (i.e., A^{-1} exists),

P2: A^{-1} is non-negative (i.e., if $a_{ij}^{-1} \geq 0$, where a_{ij}^{-1} is the (i, j) element of A^{-1}),

P3: $\|A^{-1}\|_\infty \leq \dfrac{\|r\|_\infty}{\min\limits_{i=1,2,\ldots,n} (Ar)_i}$

Prove Theorem 2 above.

This result is proved in
`https://www.tankonyvtar.hu/hu/tartalom/tkt/numerikus-modszerek-1/ch02s03.html#ssec-1-3-4`
However, this article in Hungarian (I think), but you can read the math equations. You can use this Hungarian paper as a starting point to prove Theorem 2 above. Of course, your proof must be in English.

[1]Note that it is plagiarism if you find a published solution (whether in a book or in a paper or online) and present the work as your own.