This assignment is due at the **start** of the class on Thursday, 14 November 2019.

For the questions that require you to write a MatLab program, hand-in the program and its output as well as any written answers requested in the question. Your program and its output, as well as your written answers, will be marked. Your program should conform to the usual CS standards for comments, good programming style, etc.

When first learning to program in MatLab, students often produce long, messy output. Try to format the output from your program so that it is easy for your TAs to read and to understand your results. For example, if you are asked to print a table of values, print them as a table that fits on one page. To this end, you might find it helpful to read "A short description of fprintf" on the course webpage

`http://www.cs.toronto.edu/~krj/courses/336/`

Marks will be awarded for well-formatted, easy-to-read output.

Also, your TAs will appreciate your using a word processor to write the answers to questions (or parts of questions) that do not require a program. If you do write those answers by hand, make sure that they are easy to read.

1. [10 marks; 5 marks for each part]

   Consider the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 1 & -4 & 5 & 4 \\ 1 & 0 & 2 & 0 \\ 2 & -2 & 4 & 2 \\ -1 & 0 & 1 & -2 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 5 \end{pmatrix}$$

   (a) Compute the LU factorization of A.

   That is, find

   - a permutation matrix P,
   - a lower-triangular matrix L with ones on the diagonal and $|L_{i,j}| \leq 1$ for all $i > j$
   - an upper triangular matrix U

   such that $PA = LU$.

   Show all your work.

   (b) Use the LU factorization of A from part (a) to solve $Ax = b$.

   Show all your work.

2. [10 marks: 5 marks for each part]

   (a) Do problem 2.3 and pages 99 and 100 of Heath's new textbook. Exactly the same problem is on pages 100 and 101 of Heath's old textbook

   If you don't have a copy of Heath's textbook, you can find a copy of pages 100–102 of Heath's old textbook on the course webpage:

   `http://www.cs.toronto.edu/~krj/courses/336/`

   Your textbook says to "use a library routine to solve the system of linear equations for the vector f of member forces". Instead, use the MatLab backslash operator \ to solve the linear system of equations $Af = b$ that arises in this problem for the vector f of member forces.

   Hint: read "help mldivide" in MatLab.

   Suggestion: you might find it easiest to hard-code the matrix, A, and right side vector, b, into your MatLab program. If you choose this option, you may want to start by using the MatLab function "zeros" (read "help zeros" in MatLab) to construct a matrix and a vector with all elements equal to zero and then change the appropriate elements to their nonzero values.

   Print the solution vector f. Hand in both your program and its output.

   (b) Let $\hat{f}$ be the computed solution of the linear system $Af = b$ from part (a). Explain how you can bound the relative error, $\|\hat{f} - f\|/\|f\|$, associated with the computed solution $\hat{f}$ in terms of the condition number of the matrix A associated with the linear system $Af = b$ and the relative residual, $\|r\|/\|b\|$, where $r = b - A\hat{f}$ is the residual associated with the computed solution $\hat{f}$.

   For the problem in part (a), how large is your bound on $\|\hat{f} - f\|/\|f\|$? (You might want to compute this bound in your program for part (a).)

   Hint: read "help cond" in MatLab and make sure that the norm associated with the condition number that you use agrees with the vector norm that you use for the relative error and the relative residual.

Hand in your program, its output and your written answers to the questions above.

3. [10 marks]

Do problem 2.6 on pages 100 and 101 of Heath's new textbook. Exactly the same problem is on pages 101 and 102 of Heath's old textbook

If you don't have a copy of Heath's textbook, you can find a copy of pages 100–102 of Heath's old textbook on the course webpage:

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Your textbook says to use a library routine for Gaussian elimination to solve the problem. Instead, you should use MatLab to solve this problem. In particular, use the MatLab backslash operator \ to solve the linear system of equations $Hx = b$, where H is the Hilbert matrix.

Read "help mldivide" and "help hilb" in MatLab.

Use the MatLab function "cond" for the condition number estimator.

Read "help cond" in MatLab.

Since you are asked to use the ∞-norm for the vectors in this question, make sure that you use the condition number estimator associated with the ∞-norm also.

Your textbook asks, "How large can you take n before the error is 100 percent?" The n here is the dimension of the Hilbert matrix, H (i.e., $H = \text{hilb}(n)$). This question is not very well-defined. So, instead, determine the smallest value of the positive integer n for which the relative error in the solution, $\|x - \hat{x}\|/\|x\|$, is greater than or equal to 1. To support your answer, print a table of values for $n = 2, 3, \ldots$, where each line of your table should contain n and the associated $\|x - \hat{x}\|/\|x\|$ for the problem of size n (i.e., the problem with $H = \text{hilb}(n)$). The last line of your table should be the only line for which $\|x-\hat{x}\|/\|x\| \geq 1$. That is, all of the lines before the last line should have $\|x - \hat{x}\|/\|x\| < 1$.

Your textbook also asks you to "try to characterize the condition number as a function of n". To do this, you might find it helpful to plot n versus the log of the condition number of $\text{hilb}(n)$, for $n = 2, 3, \ldots, 13$. What does this tell you about the relationship between n and the condition number of $\text{hilb}(n)$? Try to be as quantitative as possible. That is, try to find a function $f(n)$ such that

$$\text{the condition number of hilb}(n) \approx f(n)$$

Your plot should support your answer. That is, your plot should convince your TA that your function, $f(n)$, is a good approximation to the condition number of $\text{hilb}(n)$ or equivalently that the log of your function, $\log(f(n))$, is a good approximation to the log of the condition number of $\text{hilb}(n)$. (You might find using $\log_{10}$ better than $\log_e$ here.)

Finally, your textbook asks you "as n varies, how does the number of correct digits in the components of the computed solution relate to the condition number of the matrix” (i.e., hilb(n))? Again, try to be as quantitative as possible. Print a table of appropriate values to support your answer.

Hand in your programs, their output and your answers to all the questions above.