This assignment is due at the **start** of your lecture on Thursday, 24 October 2019.

Make sure you put your name and student number on the first page of your assignment and staple all the pages together.

Your TAs will appreciate your using a word processor to write the answers to questions that do not require a program (which are all the questions on this assignment). If you do write the answers by hand, make sure that they are easy to read.

1. [5 marks]

   We showed in class that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$, $\|x\|_\infty \leq \|x\|_2$.

   Show also that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$, $\|x\|_2 \leq \|x\|_1$.

   Thus, we have shown the result stated on page 54 of Heath's old[1] textbook and on page 53 of Heath's new[2] textbook (and also mentioned in class) that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$, $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

2. [5 marks]

   We also showed in class that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$, $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$.

   Show also that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$, $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

   Thus, we have shown the result stated on page 54 of Heath's old textbook and on page 53 of Heath's new textbook (and also mentioned in class) that, for any integer $n \geq 1$ and any vector $x \in \mathbb{R}^n$,

$$\|x\|_1 \leq \sqrt{n}\|x\|_2, \qquad \|x\|_2 \leq \sqrt{n}\|x\|_\infty, \qquad \text{and} \qquad \|x\|_1 \leq n\|x\|_\infty.$$

[1] I will refer to

Michael T. Heath, *Scientific Computing: An Introductory Survey*, 2nd edition, McGraw Hill, 2002.

as Heath's *old* textbook.

[2] I will refer to

Michael T. Heath, *Scientific Computing: An Introductory Survey*, Revised Second Edition, SIAM, 2018.

as Heath's *new* textbook.

I mentioned in class that vector norms always satisfy a property called *norm equivalence*. That is,

> For any two vector norms $\|x\|_a$ and $\|x\|_b$ defined for $x \in \mathbb{R}^n$, there are positive real constants c_n and C_n such that
>
> $$c_n\|x\|_a \le \|x\|_b \le C_n\|x\|_a$$
>
> holds for all $x \in \mathbb{R}^n$.

The constants c_n and C_n may depend on n, but not x.

Matrix norms also satisfy a similar norm equivalence property.

Notice that, if we combine the inequalities from Questions 1 and 2 above we have several examples of norm equivalence:

$$\begin{aligned}
\|x\|_\infty \le \|x\|_2 &\le \sqrt{n}\|x\|_\infty \\
\frac{1}{\sqrt{n}}\|x\|_2 \le \|x\|_\infty &\le \|x\|_2 \\
\|x\|_2 \le \|x\|_1 &\le \sqrt{n}\|x\|_2 \\
\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 &\le \|x\|_1 \\
\|x\|_\infty \le \|x\|_1 &\le n\|x\|_\infty \\
\frac{1}{n}\|x\|_1 \le \|x\|_\infty &\le \|x\|_1
\end{aligned}$$

The discussion above on this page is just for your information. You don't have to answer any questions about it.

3. [5 marks]

   Assume x and y are vectors in $\mathbb{R}^2$. Is it possible to have $\|x\|_1 > \|y\|_1$ and $\|x\|_2 < \|y\|_2$?

   If so, give an example.

   If not, explain why not.

4. [5 marks]

   Do question 4 on last year's midterm test.

   You can find last year's midterm test on the course webpage:
   `http://www.cs.toronto.edu/~krj/courses/336/`

5. [5 marks]

Assume A is a real $n \times n$ matrix and $\|A\|$ is a matrix norm subordinate to a vector norm. That is,

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

where $x \in \mathbb{R}^n$ is a real vector of dimension n and both $\|x\|$ and $\|Ax\|$ are the same vector norm associated with the matrix norm $\|A\|$.

A useful result is

> If $\|A\|$ is a matrix norm subordinate to a vector norm and $\|A\| < 1$, then $I - A$ is nonsingular.

One way to prove the result above is to recall that,

(a) $I - A$ is singular if and only if there exists a non-zero vector $v \in \mathbb{R}^n$ (i.e., $v \neq 0$) such that $(I - A)v = 0$, and then show that

(b) if $\|A\|$ is a matrix norm subordinate to a vector norm and $\|A\| < 1$, then there does not exist a non-zero vector $v \in \mathbb{R}^n$ (i.e., $v \neq 0$) such that $(I - A)v = 0$.

Show that point (b) above is true.

6. [5 marks]

I mentioned in class that, if

(a) A and $\hat{A}$ are both nonsingular,

(b) $x \neq 0$,

(c) $Ax = b$,

(d) $\hat{A}\hat{x} = \hat{b}$

(e) $E = \hat{A} - A$,

(f) $\Delta x = \hat{x} - x$,

(g) $\Delta b = \hat{b} - b$,

then

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right) \tag{1}$$

I got this bound from the bottom of page 60 of the Heath's old textbook. However, it turns out that this bound does not always hold. (Even textbooks contain errors!)

Give an example that shows that (1) does not always hold. That is, specify values for A, x, b, $\hat{A}$, $\hat{x}$, $\hat{b}$ that satisfy points (a)–(g) above and show that, for your specified values of A, x, b, $\hat{A}$, $\hat{x}$, $\hat{b}$, (1) does not hold. That is, for your specified values of A, x, b, $\hat{A}$, $\hat{x}$, $\hat{b}$, show that

$$\frac{\|\Delta x\|}{\|x\|} > \operatorname{cond}(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right)$$

Heath's new textbook contains a revised bound at the bottom of page 59. If you evaluate that bound for $t = 1$, you get

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right) + \mathcal{O}(1) \tag{2}$$

However, the $\mathcal{O}(1)$ in this bound makes it almost useless in practice, since $\mathcal{O}(1)$ hides an arbitrarily large constant.

A correct *useful* bound is

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\operatorname{cond}(A)}{1 - \operatorname{cond}(A)\frac{\|E\|}{\|A\|}} \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right) \tag{3}$$

which holds provided that $\operatorname{cond}(A)\frac{\|E\|}{\|A\|} < 1$. You can find a derivation of the bound (3) in Demmel's book *Applied Numerical Linear Algebra* as well as several other books on Numerical Linear Algebra. If you'd like to see Demmel's proof, take a look at

`http://www.cs.toronto.edu/~krj/courses/336/Applied_Numerical_Linear_Algebra.James_Demmel.Page_33.pdf`

He uses $\kappa(A)$ for $\operatorname{cond}(A)$, δA for E, δx for Δx and δb for Δb.