THE BEAUTIFUL GEOMETRY OF APERIODIC TESSELLATIONS

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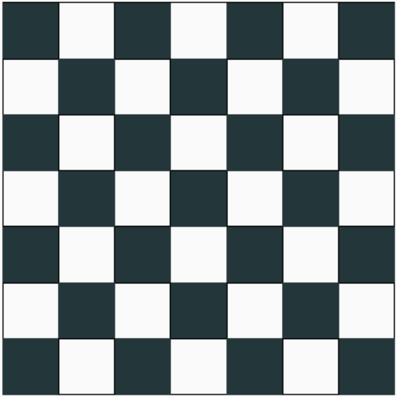
WHAT IS A TESSELLATION?

A SIMPLE TESSELLATION



Checkerboard Tessellation

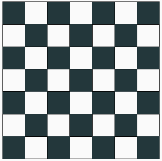
Definition:

*A **tessellation** $\mathcal{T}$ of the space $\mathbb{E}^n$ is a countable family of closed sets, T, called* tiles*:*

$$\mathcal{T} = \{T_1, T_2, \ldots\}$$

such that

1. *$\mathcal{T}$ has **no overlaps**: $\mathring{T}_i \cap \mathring{T}_j = \emptyset$ if $i \neq j$*
2. *$\mathcal{T}$ has **no gaps**: $\bigcup_{i=1}^{\infty} T_i = \mathbb{E}^n$*



COMPONENTS OF A TESSELLATION

Definition:

Let $\{T_1, T_2, \ldots\}$ be the set of tiles of tessellation $\mathcal{T}$, partitioned into a set of equivalence classes by ***criterion*** *$\mathcal{M}$. The set, $\mathcal{P}$, of representatives of these equivalence classes is called the* ***protoset*** *for $\mathcal{T}$ with respect to $\mathcal{M}$.*

Example:

Criterion: $\mathcal{M} = \{$*Colour of the tile. Only opposite colours may touch.*$\}$

Protoset: $$\mathcal{P} = \{ \blacksquare, \square \}$$

PROTOSETS ADMIT TESSELLATIONS

Definition:

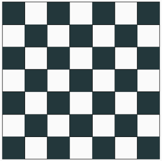
If $\mathcal{T}$ is a tessellation with protoset $\mathcal{P}$, then we say that $\mathcal{P}$ admits $\mathcal{T}$.

Example:

We say

$$\mathcal{P} = \{ \blacksquare , \square \}$$

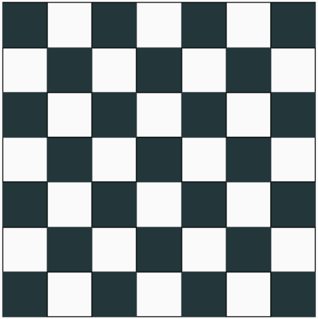
admits



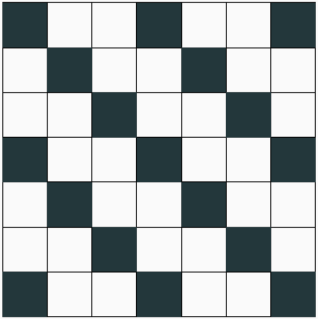
PROTOSETS CAN ADMIT MULTIPLE TESSELLATIONS

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both

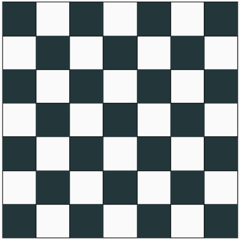


and

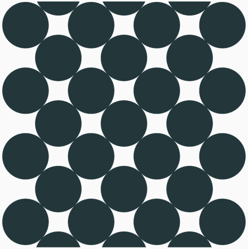


Edge deformations can force matching rules.

$\mathcal{P} = \{ \blacksquare , \square \}$ $\qquad$ $\mathcal{P} = \{ \bullet ,$  $\}$



$\rightarrow$



Matching Rules $\implies$ Deformed Protoset

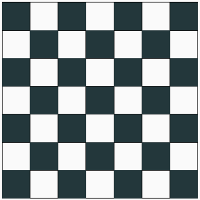
DESCRIBING TESSELLATIONS

Definition:

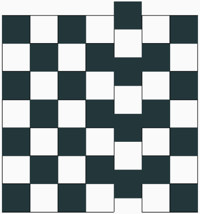
A tessellation is said to be ***periodic*** *if it admits translational symmetry in two directions.*

Definition:

A tessellation is said to be ***non-periodic*** *if it admits no translational symmetries.*



Periodic



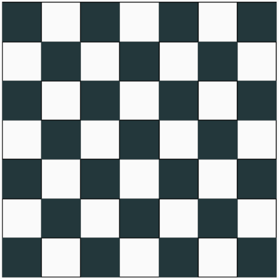
Non-Periodic

THE APERIODIC QUESTION

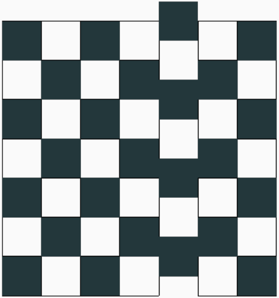
Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.



Periodic



Non-Periodic

THE APERIODIC QUESTION

Checkerboard protoset,

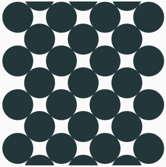
$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.

Some protosets,

$$\mathcal{P} = \{ \bullet , \square \}$$

admit only **periodic** tessellations.



Checkerboard protoset,

$$\mathcal{P} = \{ \blacksquare , \square \}$$

admits both **periodic** and **non-periodic** tessellations.

Some protosets,

$\mathcal{P} = \{ \bullet ,$  $\}$

admit only **periodic** tessellations.

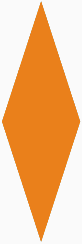
Are there any protosets,

$$\mathcal{P} = \{?\}$$

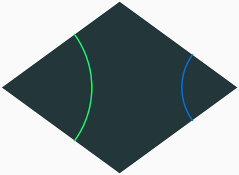
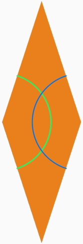
that admit only **non-periodic** tessellations?

AN APERIODIC TESSELLATION

THE PENROSE RHOMBS PROTOSET

$\mathcal{P} = \{$  ,  $\}^*$

$*$ With matching rules

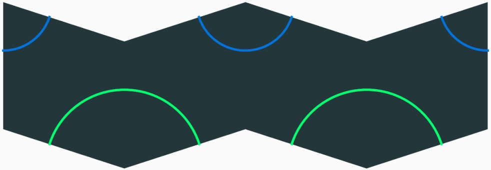
$\mathcal{P} = \{$  ,  $\}$

admits only **non-periodic** tessellations, called Penrose Tessellations.

Cannot be constructed through local procedures.

Example:

This arrangement

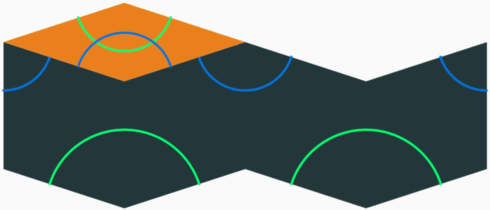


follows the matching rules.

Cannot be constructed through local procedures.

Example:

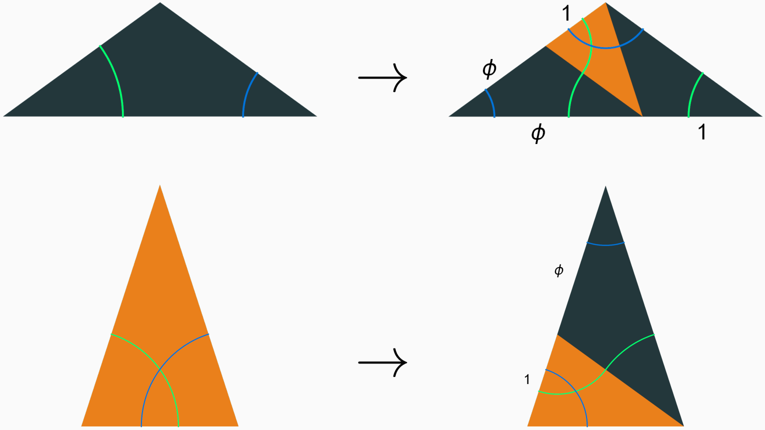
This arrangement



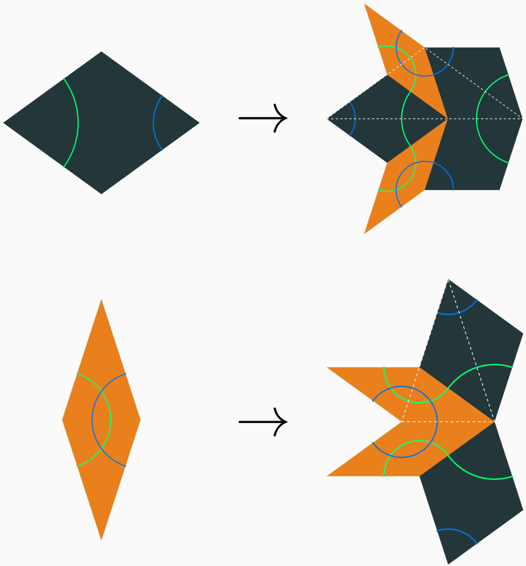
follows the matching rules, but is a mistake:

CONSTRUCTION BY SUBSTITUTION

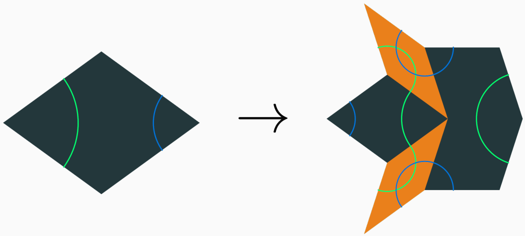
First, consider the Rhombs as joined triangles:

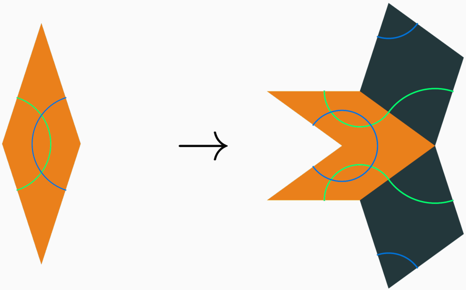


SUBSTITUTION RULES ON PENROSE RHOMBS

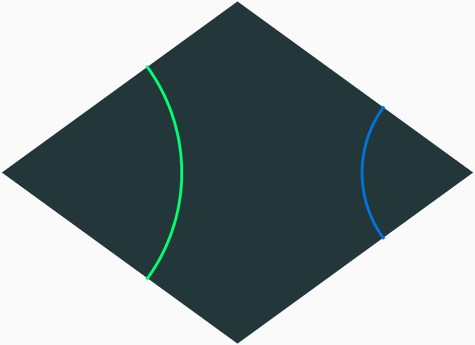


SUBSTITUTION RULES ON PENROSE RHOMBS

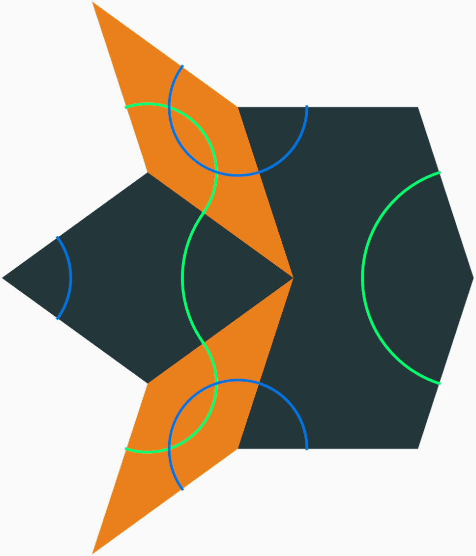




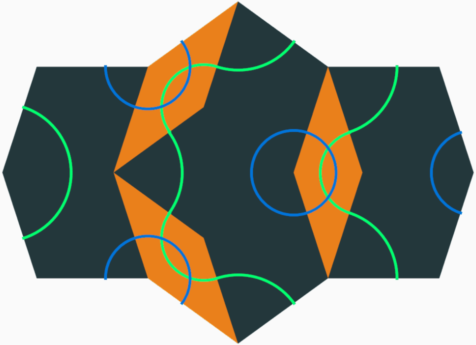
START WITH A SEED: THICK RHOMBUS



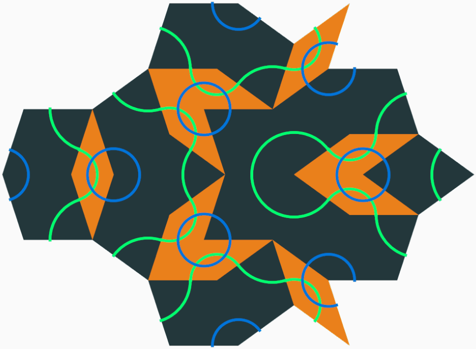
APPLY SUBSTITUTION RULES



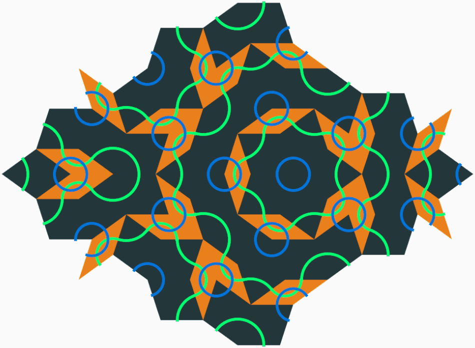
APPLY SUBSTITUTION RULES AGAIN



APPLY SUBSTITUTION RULES AGAIN AND AGAIN



APPLY SUBSTITUTION RULES AGAIN AND AGAIN AND AGAIN



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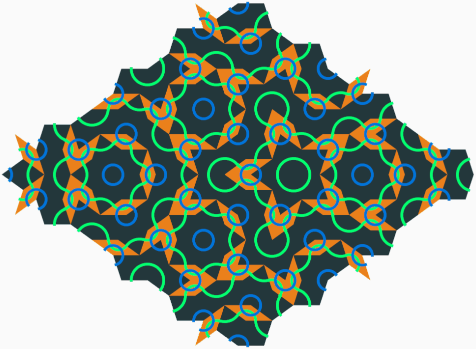
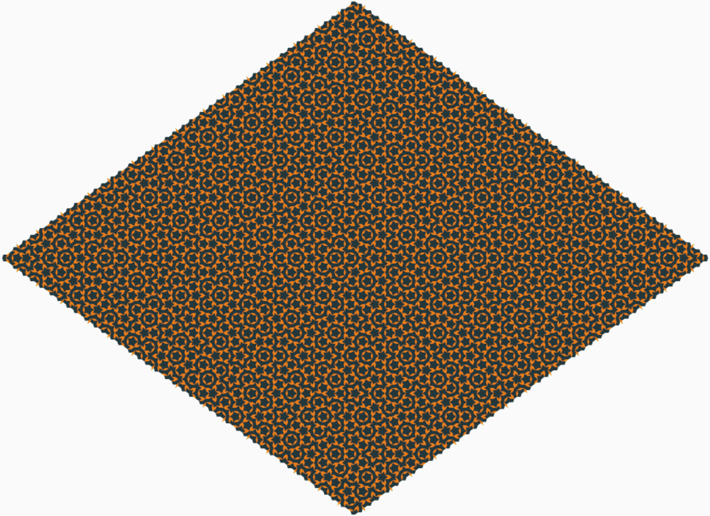


APPLY SUBSTITUTION RULES 10 TIMES

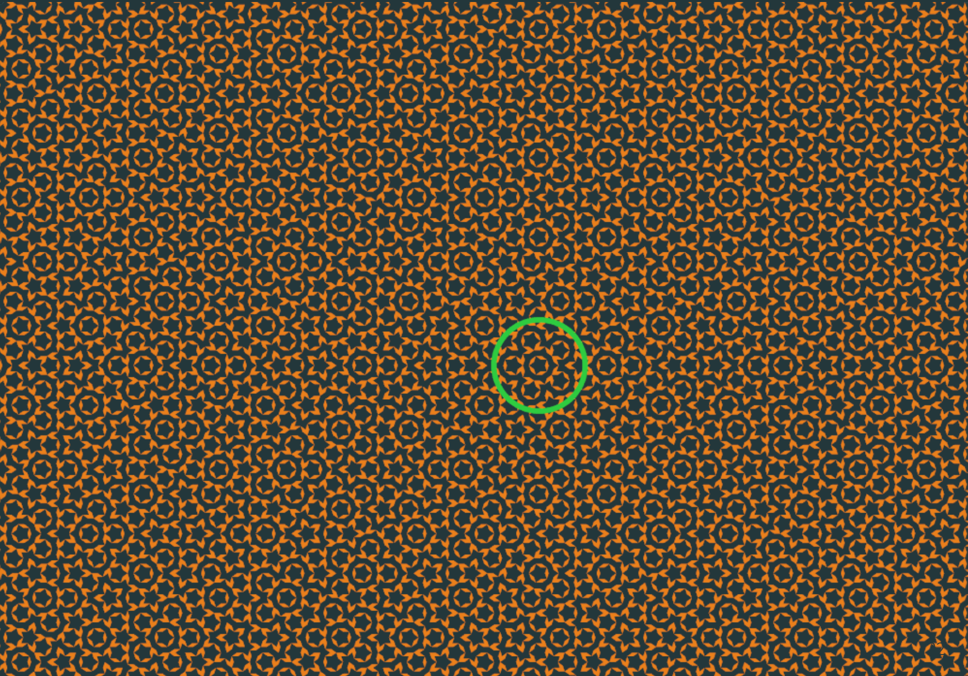


APPLY SUBSTITUTION RULES 11 TIMES

PENROSE TESSELLATION FEATURES

Any pattern with diameter d will begin repeating within $\frac{\sqrt[3]{\phi}}{2}d$ from its perimeter.

DENSITY OF PATTERNS



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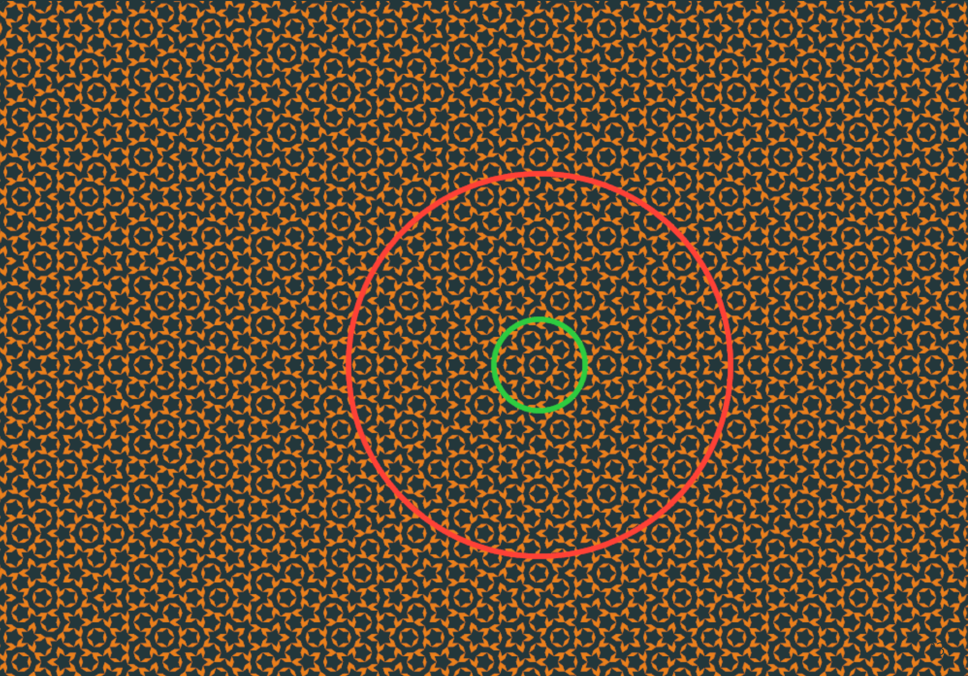

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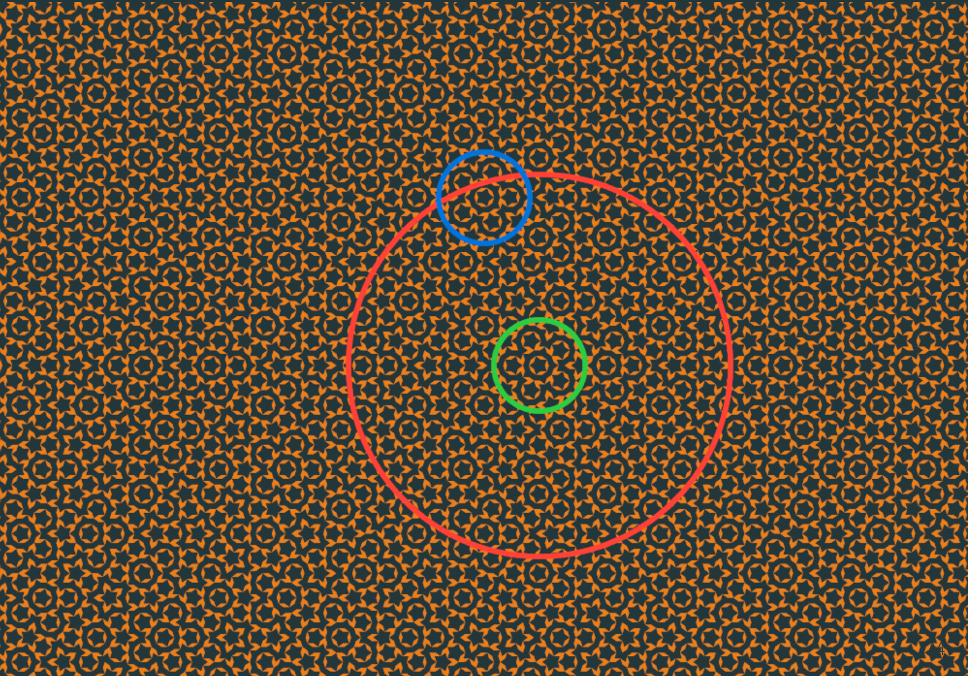

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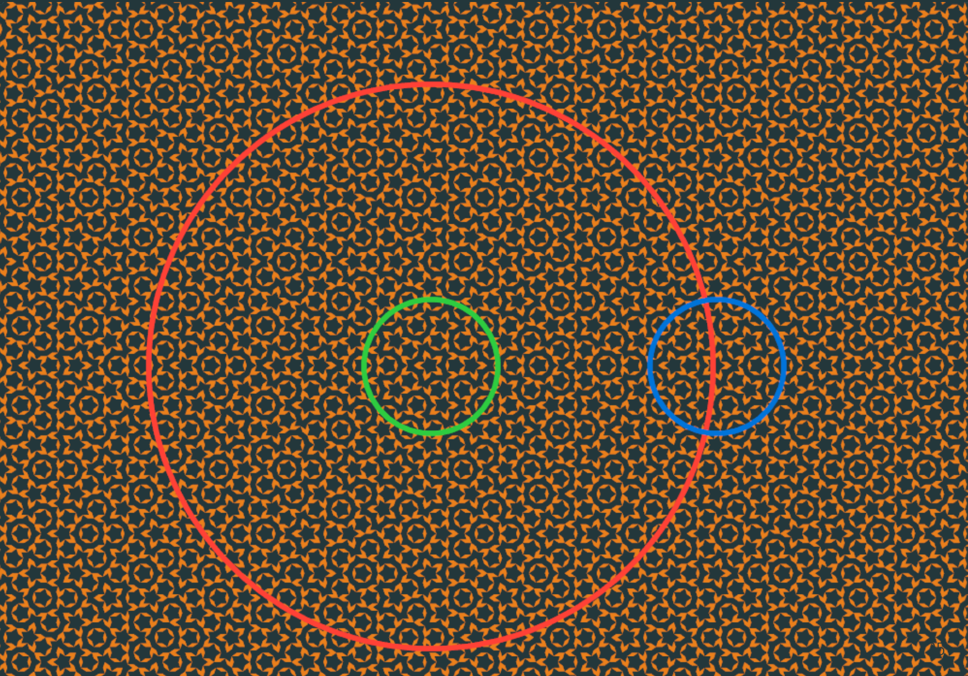
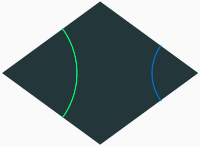
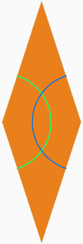


Penrose Tessellations can admit five-fold rotational symmetry.

(Impossible for periodic tessellations)

of 

$= \phi$

of 

Questions?

github.com/jessebett/PenroseTilingThesis

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