CSC303: A0

Due Thu Jan 25 at 11:59PM

This assignment covers prerequisite linear algebra & probability concepts. For review resources, see the "Linear Algebra Review" and "Probability Review" sections at `https://www.cs.toronto.edu/~ianberlot/303s24/material.html`

You will receive 20% of the points for any (sub)problem for which you write "I do not know how to answer this question." If instead you submit irrelevant, erroneous, or blank answers then you will receive 0 points. You may receive partial credit for the work that is clearly "on the right track."

A latex file of this assignment is available on Quercus.

Before you're finished with this assignment, please check the submission checklist:

1. Your name and student number are written down **on the last page**
2. Any use of generative AI has been documented, and declared (see course syllabus for the details of what is required – in particular, you should transcribe and include all your interactions with the tool)
3. Your solutions are contained within a *legible* PDF of *reasonable filesize* (I believe the MarkUs filesize limit is about 8MB this year).
4. The correct .pdf file has been uploaded to MarkUs.
5. Your submission has the exact filename requested in MarkUs, including the ".pdf" file extension.
6. Your submission can be viewed from *inside* of MarkUs. Note that if MarkUs can't preview your submission, you may be missing the ".pdf" file extension.

Question 1: (2 points) For each of the following, either evaluate the matrix multiplication (no justification required) or very briefly explain why they cannot be evaluated.

(a) [0.5 points]

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1&2&3\\1&3&4\\3&4&5\end{bmatrix}$$

(b) [0.5 points]

$$\begin{bmatrix}1&2&3\\1&3&4\\3&4&5\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

(c) [0.5 points]

$$\begin{bmatrix}1&2&3\end{bmatrix}\begin{bmatrix}1&2&3\\1&3&4\\3&4&5\end{bmatrix}$$

(d) [0.5 points]

$$\begin{bmatrix}-1&3&2\\5&-3&2\\1&0&1\end{bmatrix}\begin{bmatrix}1&2\\1&3\\3&4\end{bmatrix}$$

Question 2: (1 points) Use block matrix multiplication to solve for x_1, x_2, x_3 and x_4. **Do not** calculate the inverse of a 4×4 matrix as part of your solution.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ -3 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & -2 & 2 & 3 & 1 \\ 1 & 2 & -1 & 0 & 1 & 3 \\ -1 & 0 & x_1 & x_2 & 3 & 3 \\ -1 & -2 & x_3 & x_4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 12 & -8 & 5 & 12 & 14 \\ 1 & -1 & 2 & 0 & 4 & -1 \\ 1 & 2 & 1 & -2 & 4 & 7 \\ -9 & -7 & 4 & -5 & -5 & 3 \end{bmatrix}$$

Question 3: (2 points) Find the null space of the following matrix, and express it as a span of vectors.

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix}$$

Question 4: (1 points) What does it mean for the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k \in \mathbb{R}^n$ to be linearly independent?

Question 5: (2 points) For each of the following, state whether $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbb{R}^3$ form an orthonormal basis of $\mathbb{R}^3$. Briefly justify.

(a) [0.5 points] $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) [0.5 points] $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) [0.5 points] $\mathbf{b}_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

(d) [0.5 points] $\mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Question 6: (3 points) Let M be a square $n \times n$ matrix.

(a) [2 points] Let $\mathbf{e} \in \mathbb{R}^n$ be an eigenvector of M with eigenvalue λ. Is e an eigenvector of the matrix $M^2 = MM$? If yes, state the eigenvalue and briefly explain. If not, explain why not.

(b) [1 points] Now suppose that M is a *positive semidefinite* matrix (i.e., M is symmetric and $\forall \mathbf{x} \in \mathbb{R}^n$: $\mathbf{x}^T M \mathbf{x} \geq 0$). Prove that it is not possible for M to have an eigenvector, $\mathbf{v}$, with a negative eigenvalue. (Aside: a symmetric matrix can only have real eigenvalues; stick with real values for this question). (HINT: Remember the definition of $||\mathbf{x}||$)

Question 7: (1 points) Given events A and B with non-zero probability of occurring, define the conditional probability of B given A (i.e., define $P(B|A)$)

Question 8: (3 points) Let A and B be two events with non-zero probability.

(a) [1 points] What does it mean for A and B to be independent events?

(b) [1 points] What does it mean for A and B to be mutually exclusive events?

(c) [1 points] Give an example where A and B are both mutually exclusive events, and independent events. Briefly explain. If this is not possible, provide a proof.

Question 9: (4 points) Amir and Vered are playing a board game. This game involves an opaque bag containing 9 grey tokens, and 1 golden token. It also involves a biased 8-sided dice: the dice rolls 1, 2, and 3 with 10% probability respectively, it rolls 4, 5, 6 and 7 with 12.5% probability respectively, and it rolls a 8 with a 20% probability.

(a) [2 points] To determine who goes first, Amir rolls the 8-sided die, and Vered randomly draws that many tokens from the bag, without replacement. If Vered draws the gold token then she goes first, otherwise Amir goes first. (e.g., The following is one possible valid outcome: Amir rolls a 3. Vered takes 3 tokens out of the bag, leaving 7 tokens in the bag. Vered randomly drew a grey token, a gold token, and a grey token – since she drew the gold token she goes first). What is the probability that Vered goes first in the game? Show your work.

(b) Let X be the random variable corresponding to a roll of the die. Let's define $\mathbb{I}_{X=x}$ as a random variable that is 1 if $X = x$, and 0 otherwise (e.g., $\mathbb{I}_{X=2}$ is 1 if the dice rolls a 2, and is zero otherwise).

(i) [1 points] What is $\mathbb{E}[\mathbb{I}_{X=2}]$? Briefly explain.

(ii) [1 points] In the general case, what is $\mathbb{E}[\mathbb{I}_{X=x}]$? Briefly explain.

Question 10: (3 points) Recall the binomial distribution: $X \sim \text{binom}(n, p)$ iff $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$. Equivalently, if we have n independently and identically distributed Bernoulli variables, each of which succeeds with probability p, (i.e., $T_1, T_2, \ldots T_n \overset{\text{iid}}{\sim} \text{Bern}(p)$), then X is equal in distribution to the number of trials that succeed. In other words, if $T_1, T_2, \ldots T_n \overset{\text{iid}}{\sim} \text{Bern}(p)$, then $(\sum_{i=1}^{n} T_i) \sim \text{binom}(n, p)$.

(a) [2 points] What is the expectation and variance of $X \sim \text{binom}(n, p)$? Justify your answer. HINT: Use the Bernoulli definition of the binomial distribution

(b) [1 points] What is $P(X \geq 1)$? HINT: There is a short answer to this question, rephrasing the question may help

END OF ASSIGNMENT 0

If you are typsetting the assignment using the provided LaTeX, then please write your name and student number below.

NAME: Your name should go here, on the last page.
STUDENT NUMBER: You student number should go here, on the last page.