Social and Information Networks

University of Toronto CSC303
Winter/Spring 2023

Week 6: February 13-17

Mon. Feb 13th: Announcements & Corrections

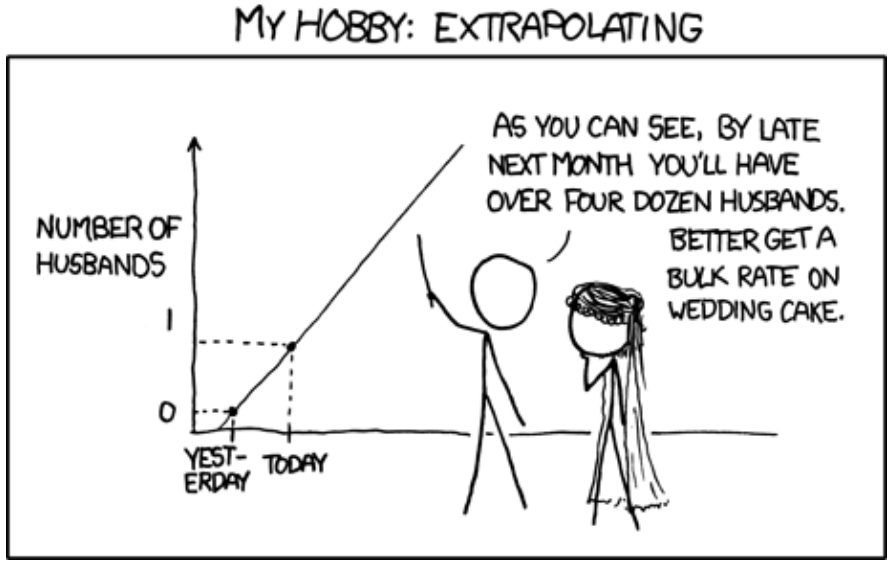
- Mid-course evaluation now available on Quercus
  - Anonymous, 5-10 minutes, open until Feb 26th
  - I really appreciate knowing how things are going, and how to do better :)

- A typo has been fixed in the course handbook, in *Guidelines about Due Dates & Missed Work*
  - The policy is unchanged
  - It's obvious from context, but for the sake of clarity: The example date given is for assignment 2, not assignment 1
    - ⋆ Please don't try submitting A1 in March ;)

- We've now got a list which we'll be populated with papers as they are claimed for the critical review
  - See the assignments tab on the course website
  - Or click here

Mon. Feb 13th: Announcements & Corrections

- Happy Valentine's Day tomorrow!



Comic from xkcd

- Assignment 1 is due this Thursday :(
  - ... but it's reading week after that :)

This week's agenda

- Power Laws
  - Definition
  - Zipf's law
  - Dynamics that often give rise to power laws
- Rich-get-richer model
- Salganik et al. music popularity experiment
- Ranking web results
  - Hubs and Authorities
  - (Scaled) Page Rank
  - Proof-sketches of convergence

Roadmap: where we have been and whats next

Chapter 20 started off with a discussion of the small worlds phenomena and an insightful understanding of how decentralized search can work.

Previously, we were led to the observation that geographical distance (or social distance) correlates with friendship such that Prob[v is a friend of u] $\approx [rank_u(v)]^{-1}$.

Furthermore, there is a sense that long distance friendships are "rare".

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This week we will be building on these ideas.

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A phenomena satisfies Zipf's Law when:

$$f \approx \frac{a}{r^c}$$

for some constants a and c

Why care about power laws?

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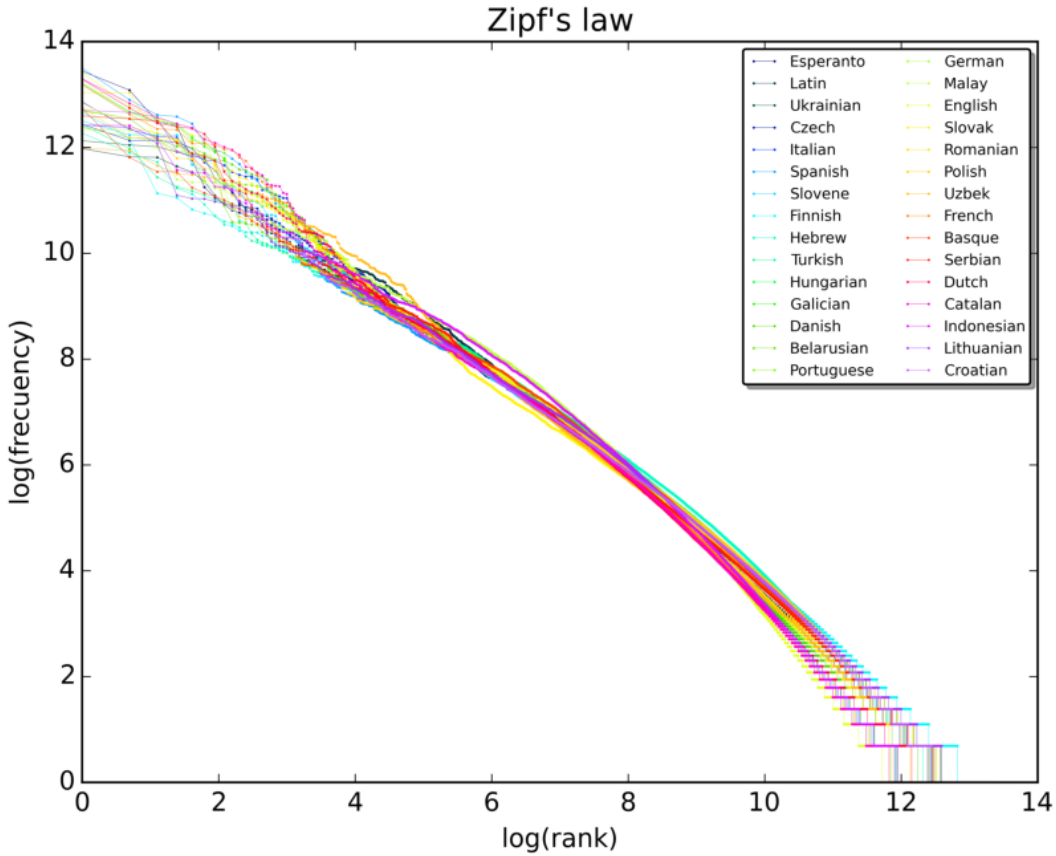
- Chapter 18 calls attention to the fact that power law distributions often occur in network and natural phenomena.
- We've already seen power laws emerge in the probability of friendship forming with respect to both distances, and ranks
- Where else do they appear?

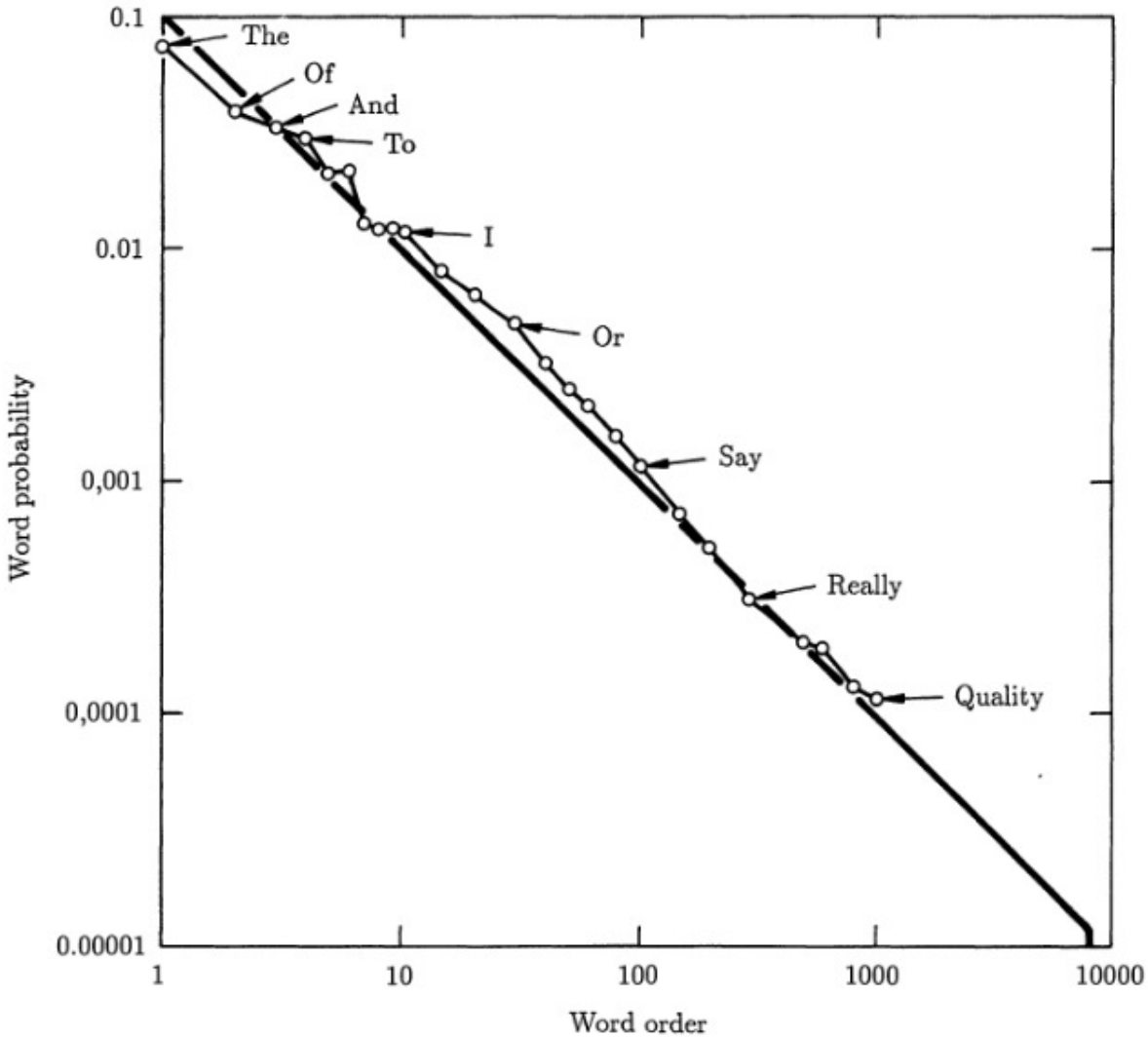
Zipf's law in text

- In any book, let w be a word within. If we calculate it's frequency f_w (i.e. number of occurrences in the text) and it's rank r_w (i.e. is the word the first, second, ... nth most common word in the book) then we find that $f_w \propto 1/r_w$ (or equivalently, $\log f_w \approx -\log r_w + C$)



[Image By SergioJimenez - Own work, CC BY-SA 4.0, link]

Power law in text



[Image from Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise by Manfred Schroeder]

Zipf's law in population centres

- Similarly, if we consider cities t and let f_t be the population of a city, and r_t be the city's rank by population, we have $f_t \approx a/r_t$

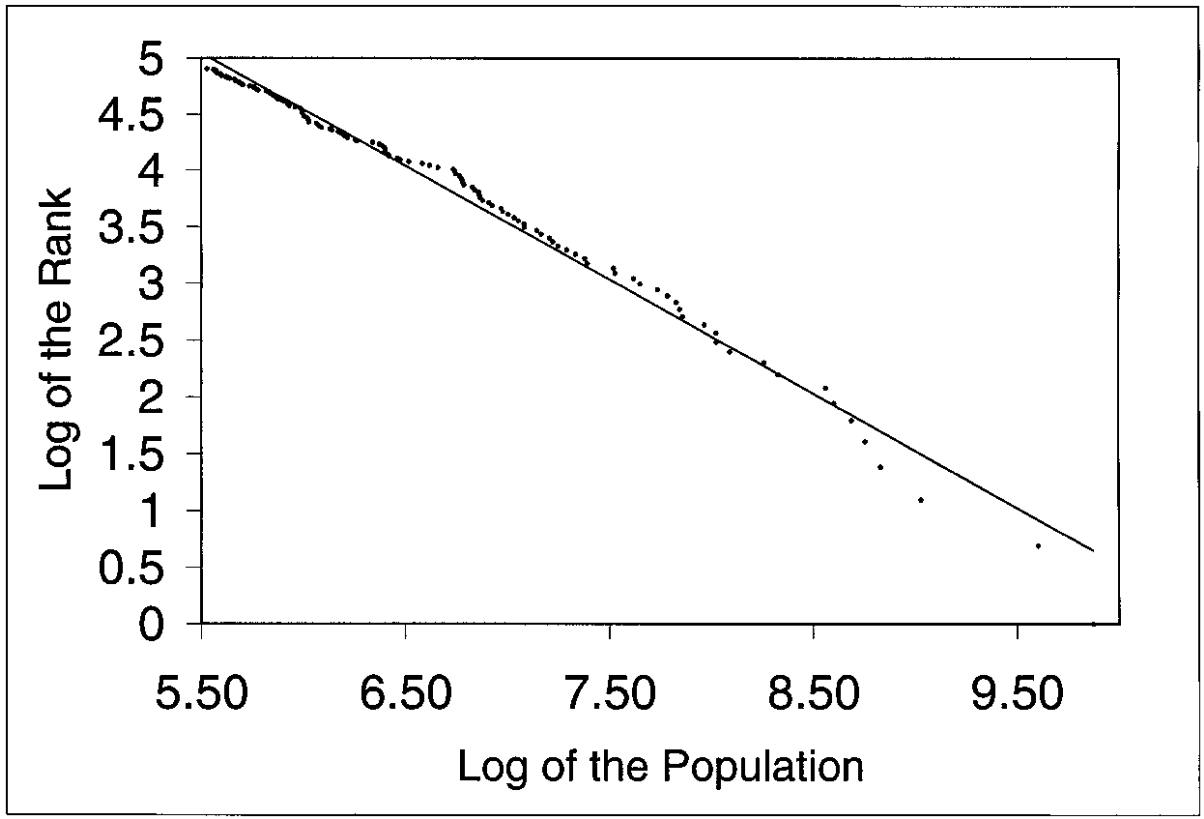


FIGURE I
Log Size versus Log Rank of the 135 largest U. S. Metropolitan Areas in 1991
Source: Statistical Abstract of the United States [1993].

[Image from Gabaix, 1999]

Power laws in websites and products

- Power laws also arise in the popularity of websites and commercial products

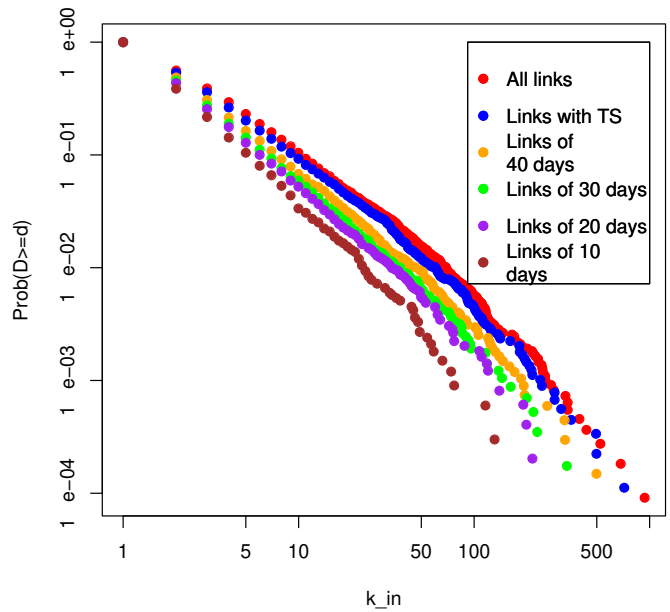


Fig. 6: *Temporal changes in the in-degree distributions in TREC.*

[Image from Shi et al.]

- Empirically, in the web network (i.e. an *information network*), the probability that a site will have k in-links is proportional to k^{-2}. (More precisely, proportional to $k^{-(2+\epsilon)}$ for some $\epsilon > 0$.

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- Consequence: Events may be less rare then they appear at first glance
- Key takeaway: extreme events (e.g., for a site to have very many in-links) is **not so rare** when compared with what would be predicted by independent decisions.

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- What if people chose where to live independent of the city? What would be (the distribution of) the population of cities?
- What if we all independently chose to read books not dependent on current events or what friends (or an online system) recommended? How rare would it be to have a huge best seller?
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As is well understood, the **Central Limit Theorem** tells us that "a quantity that can be viewed as the sum (or average) of many small independent random effects will be well-approximated" by a *normal distribution*.

The normal distribution

The normal or Gaussian distribution has the following probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

As we know, normal distributions have a *bell shaped curve*.

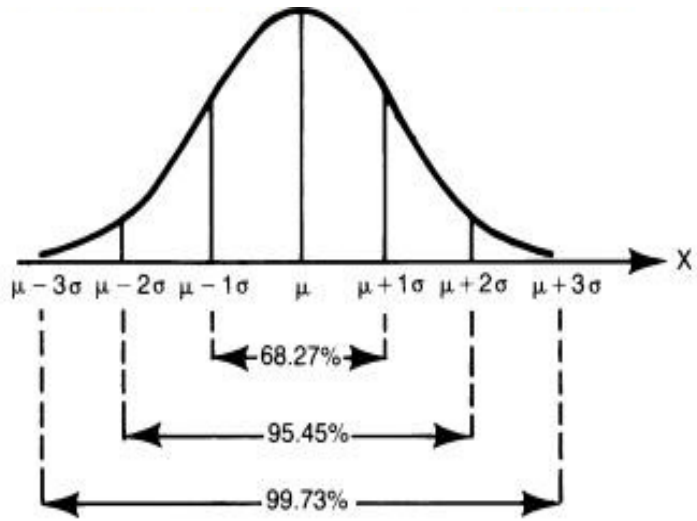

Figure 2

Percent	99.73%	99%	95.45%	95%	90%	80%	68.27%
No. of ± σ's	3.00	2.58	2.00	1.96	1.645	1.28	1.00

From: http://www.answers.com/topic/normal-distribution

So how rare is rare?

- In a normal distribution, the probability of an outlier (i.e. an exceptional event) decreases exponentially with distance from the mean
  - if say in-links followed a normal distribution, then the probability that a given site would have k links would decrease *exponentially* in k
  - Very small or large "outliers" would be highly improbable

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- One in a billion vs better than 1 in a 1000!
  - Do note that the above is not a probability as we've omitted the normalizing constants
  - However, the example illustrates the difference in the rate of decay

So where are we going?

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As we have mentioned before, one of the most fundamental questions for social networks concerns how they evolve. What is the interplay between selection and influence?

This is a difficult question! Perhaps the dynamics of information networks created by individuals can be better understood than the dynamics of friendships, political affiliations, opinion formation, etc...

We will see a network dynamic that leads to a power law distribution.

Recap

- Power Laws
  - Definition
  - Zipf's law
  - Dynamics that often give rise to power laws

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- Kumar et al [2000] proposed a preferential attachment model that can explain the power law distribution
  - Recall, the observed distribution of in-links is: Prob[a site has k in-links] $\propto k^{-(2+\epsilon)}$ for a small $\epsilon > 0$

A "rich get richer model" for in-links on the Web

Here is the model proposed in Kumar et al article:

1. Web pages are created sequentially, and named $1, 2, \ldots N$. (Of course, N keeps growing but we are looking at the web at some point in time)

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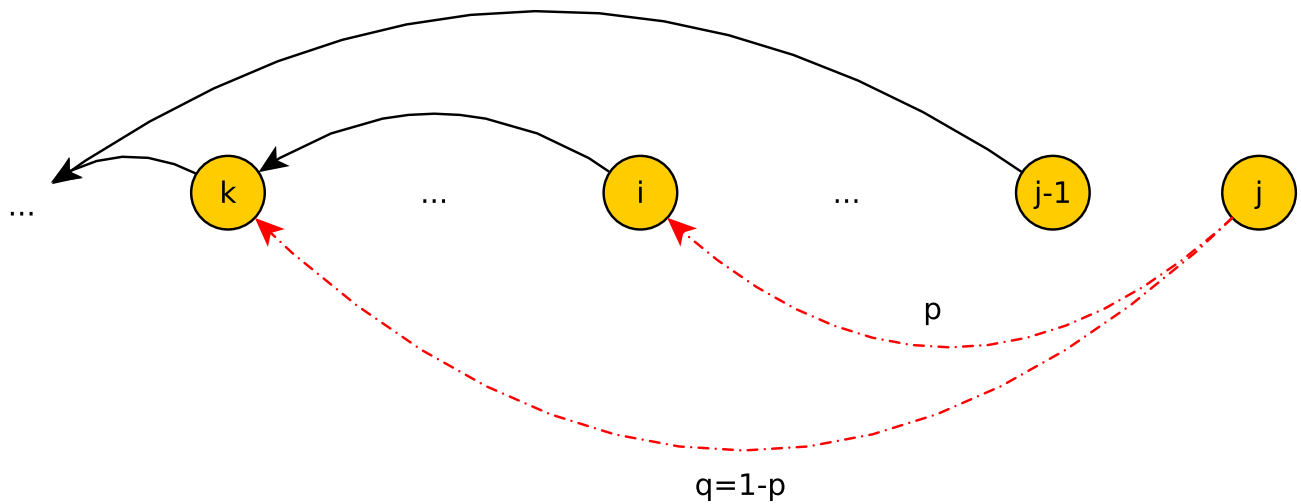

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- Our simplified model only creates one link
- Despite this, **the power law exponent does not change**
  - The key parameter is p – whether or not we're linking to a page selected uniformly at random

The linking model continued

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This is, of course, the idea behind *popularity*!

- e.g., the more people that are reading a current novel, the more likely that you might want to read it
- For various social and economic reasons why some large cities continue to grow.

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Hedge: As the text states clearly, the goal of this model is not to capture all the reasons why people create links on the Web (or links in other networks) but rather to explain why it is reasonable to expect power laws to arise from such popularity phenomena.

An informal analysis for the simplified preferential attachment model proposed for Web in-links

A precise analysis of even the simple one link per page preferential attachment model is technical.
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Specifically, we'll consider a continuous deterministic variable $x_\ell(t)$, that approximates the discrete random variable $X_\ell(t)$, the number of in-links to a page ℓ at time $t \geq 0$.

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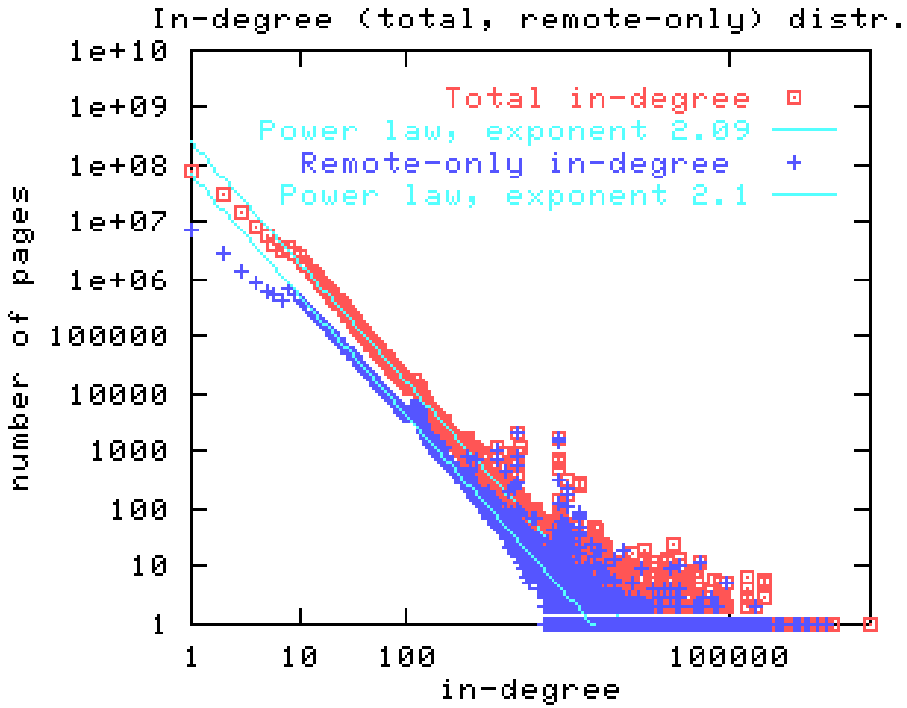
$$\frac{dx_\ell}{dt} = p\frac{1}{t} + q\frac{x_\ell(t)}{t}$$

- From some basic calculus (see Ch 18.7) this leads to a power law distribution proportional to k^{-c} with $c = 1 + 1/q$

The deterministic continuous model



[Fig 18-2 in E&K]

- As $p \to 0$, the exponent $c = 1 + 1/q$ limits to the observed exponent $c = 2 + \epsilon$ for the observed in-link power law distribution
- As $p \to 1$, the exponent limits to ∞ making a large number of in-links very unlikely.

Aside: Open Questions

- It's worth noting that although the preferential attachment model suggests that popularity phenomena lead to power laws, it still cannot explain all the examples we saw
- $c = 1 + 1/q \in [2, \infty)$, yet in text and cities the observed exponent is 1

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- It's worth noting that although the preferential attachment model suggests that popularity phenomena lead to power laws, it still cannot explain all the examples we saw
- $c = 1 + 1/q \in [2, \infty)$, yet in text and cities the observed exponent is 1

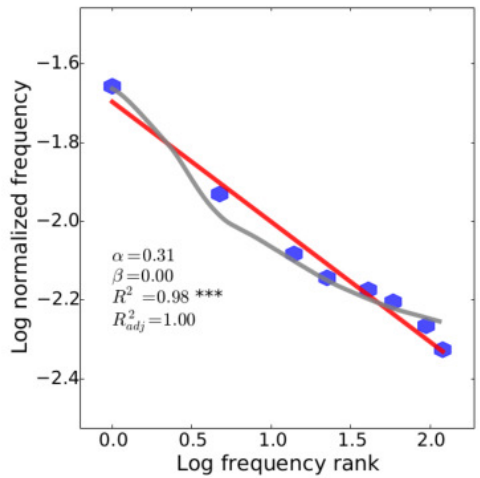

Figure 8: An approximate power law distribution of novel alien names used by subjects in making up a story.

[Figure from Piantadosi (2014)]

- Piantadosi (2014)'s prompt was "An alien space ship crashes in the Nevada desert. Eight creatures emerge, a Wug, a Plit, a Blicket, a Flark, a Warit, a Jupe, a Ralex, and a Timon. In at least 2000 words, describe what happens next"; observed is $c = 0.31$

Sensitivity to unpredictable initial stages in network dynamics

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  - Clearly, the initial stages of a dynamic process are critical and that is why advertising, promotions, etc. are so important

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  - if we could replay history many times, we would expect the resulting distribution to be the same
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  - But would the same books, the same movies, the same pop stars, the same web pages, etc continue to be the most popular?
- Intuition suggests there is considerable "luck" in exactly who or what becomes popular; yet we also believe that "quality" is also important
- How can we rewind history", to try and find out?

An experiment to “rewind history”

Although we can’t rewind history, Salganik et al perform an interesting experiment (in fact, two experiments at different times with different participants) to observe the impact of the initial random stages in a dynamic process. (the article is available on the course website).

The Salganik et al experiment

Here is their experiment:

- They created 9 copies of a music streaming site with 48 "obscure" (as determined by some experts) songs of varying "quality"
- Approximately 7200 young participants were recruited to listen to the music, knowing only the name of the band and the song.
- In each of the copies, participants sequentially listened to some music selections, rated the music and then were given the opportunity to download copies of songs they liked.

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- The experiment was then repeated, with the 8 site copies displaying songs sorted by downloads instead of randomly

The findings in the Salganik experiment

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As the text points out in section 18.6, how recommendation systems are designed can impact how people make choices, leading to increased "rich get richer" phenomena, or alternatively exposing people to less popular items.

Visualizing the long tail of a power law distribution

- Once we accept a power law nature of popularity, it is instructive to consider the consequences for a given industry
  - Consider that the nature of the sales curve that would be dictated by a power law distribution

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  - Consider that the nature of the sales curve that would be dictated by a power law distribution

The shape of the long tail in a power distribution raises the question as to how many sales can be obtained from less popular (e.g. niche items).

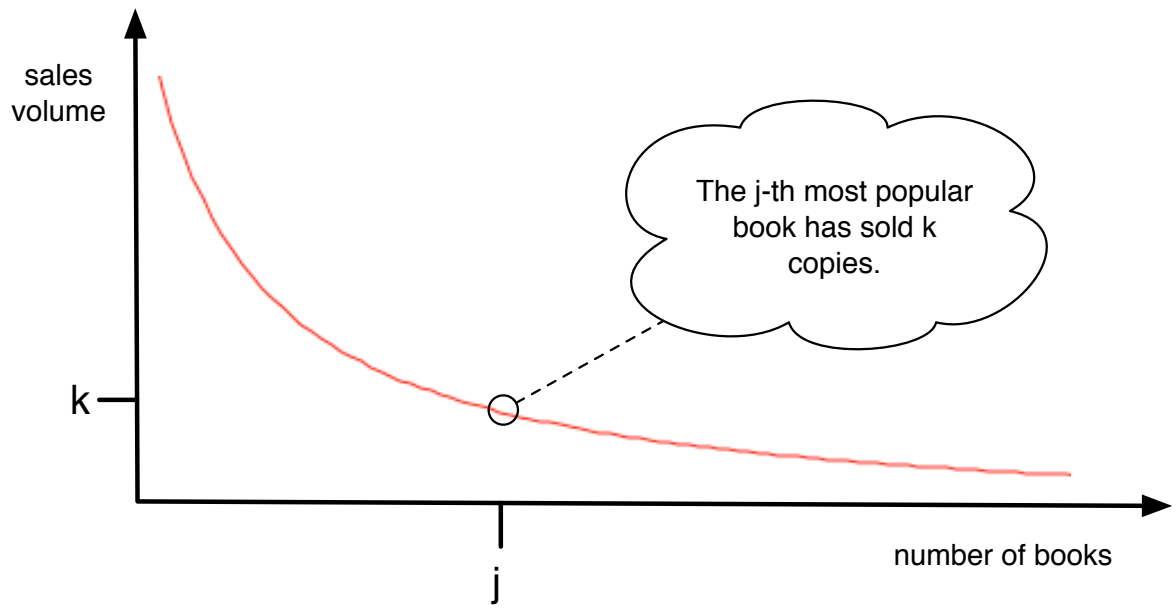

Figure: [Fig 18-4 in E&K] text; how many copies of the j^{th} most popular items have been sold.

Recap

- Power Laws
  - Definition
  - Zipf's law
  - Dynamics that often give rise to power laws
- Rich-get-richer model
- Salganik et al. music popularity experiment

Wed. Feb 15th: Announcements & Corrections

- Taking another look at the Salganik et al. paper, they're a bit vague, but I believe ratings were not shared – only downloads were explicitly stated to be visible
- A1 due tomorrow
- Critical review groups & paper choices due by March 3rd (email me)
- Midterm will be March 10th
  - Covers up to wherever we finish today
- Office hours today after class (Zoom + in person), and at 10 PM (Zoom only)
- Reading week next week
  - Try and get some rest! I've tried to avoid making things due right after reading week :)
  - No regularly scheduled office hours in reading week; but I'm available by appointment – email me :)

Search and ranking on the Web

Our next topic is to understand how the popularity of a web page is determined and how that impacts its rank in the responses to a query.

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However, we do have a basic idea as to how these search engines rank sites given a query. In fact, at the most elementary level, the main idea is an old one.

Aside: In the 1960s and 70s, there was a basic argument as to whether online search and ranking was a more or less normal algorithmic search and optimization problem or one that required "intelligence" (i.e. the ability to understand natural language). Who won this argument?

Search and ranking of Web documents; the role of link popularity

The most basic approach is to treat a document as a bag of words and then use "normalized" word counts (and pairs,triplets of words) to identify and rank documents relating to the query. This became enhanced by more sophisticated contextual aspects of word occurrences, etc and today machine learning algorithms are also used in classifying a search query.

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But early in the development of popular search engines, a popularity aspect was added where the ranking of a document also depended on the link structure and the popularity of a Web page in the Web network (or at least in that part that seems relevant to the query).

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Two algorithms were independently proposed for determining the popularity of a Web page, namely Hubs and Authorities developed at IBM, and Page Rank, developed and integrated into Google's search engine.

Link analysis and page popularity

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We will begin with the Hubs and Authorities ranking algorithm and then the Page Rank algorithm.

Hubs and Authorities

- A simple way to utilize links to rank web pages would be to think of each link from A to B as an endorsement or vote by A for B.

- Question: Assuming it's tractable, then what's wrong with just counting the number of in-links?

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  - if Wayne Gretzky has a web site with a link suggesting where to buy hockey equipment, that should be more meaningful than my recommendation about hockey equipment

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  - if Wayne Gretzky has a web site with a link suggesting where to buy hockey equipment, that should be more meaningful than my recommendation about hockey equipment

  Spoiler alert: I don't play or watch hockey

Reinforcement of Hubs and Authorities.

- This then becomes the motivation (and seemingly circular reasoning) behind hubs and authorities:
  - The best "authorities" on a subject (places to buy equipment) are being endorsed by the best "hubs" (people who know where to buy equipment)
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- Comment: The word "authority" is not generally an accurate way to describe high ranking documents. These might better be referred to (barring other information) as the most relied upon sites. This is also different from "the most popular" sites which might better be measured in terms of the number of clicks being received. Hubs then are the most reliable endorsers.

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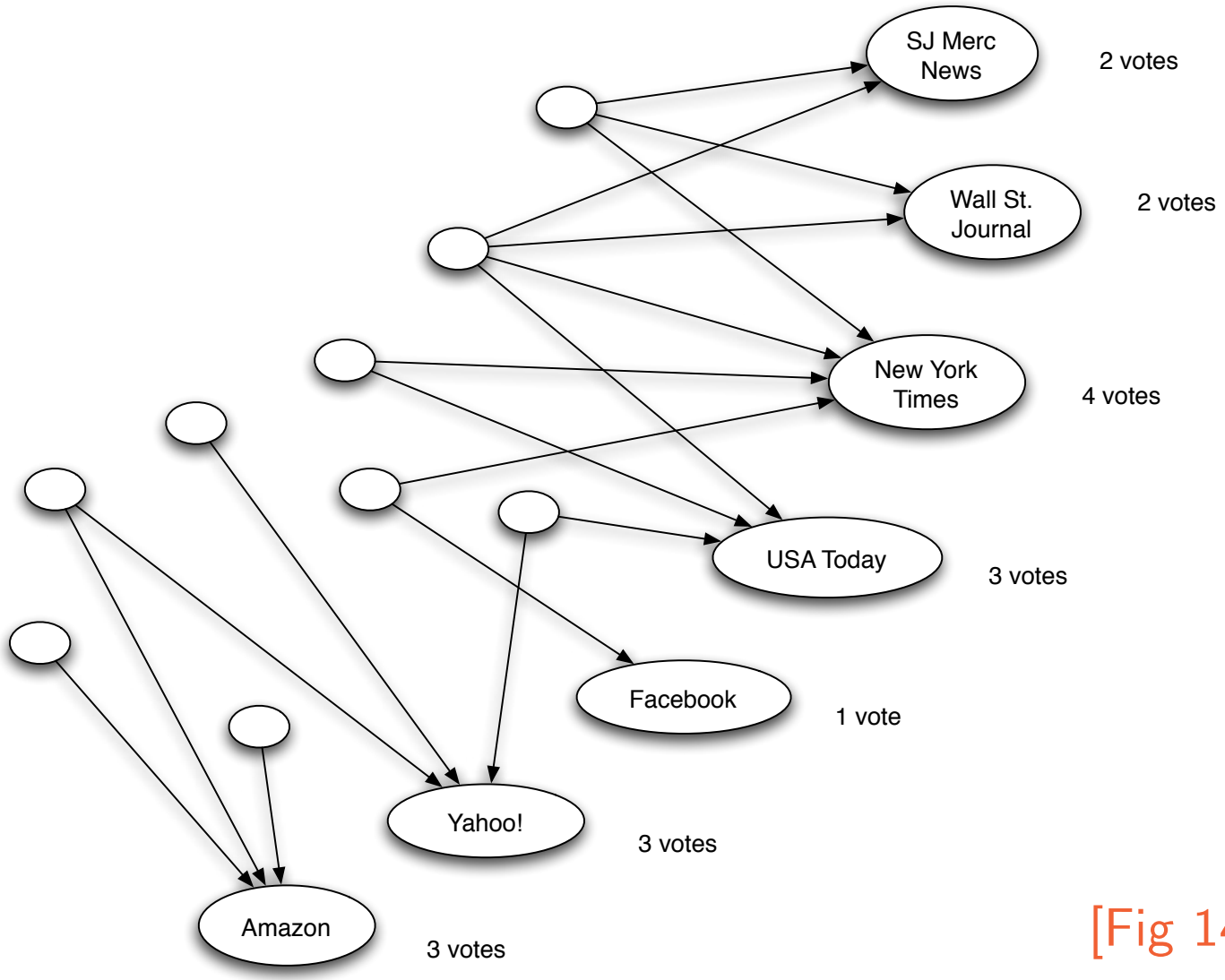
    3. Normalize so that sum of A weights is 1 and sum of H weights is 1.

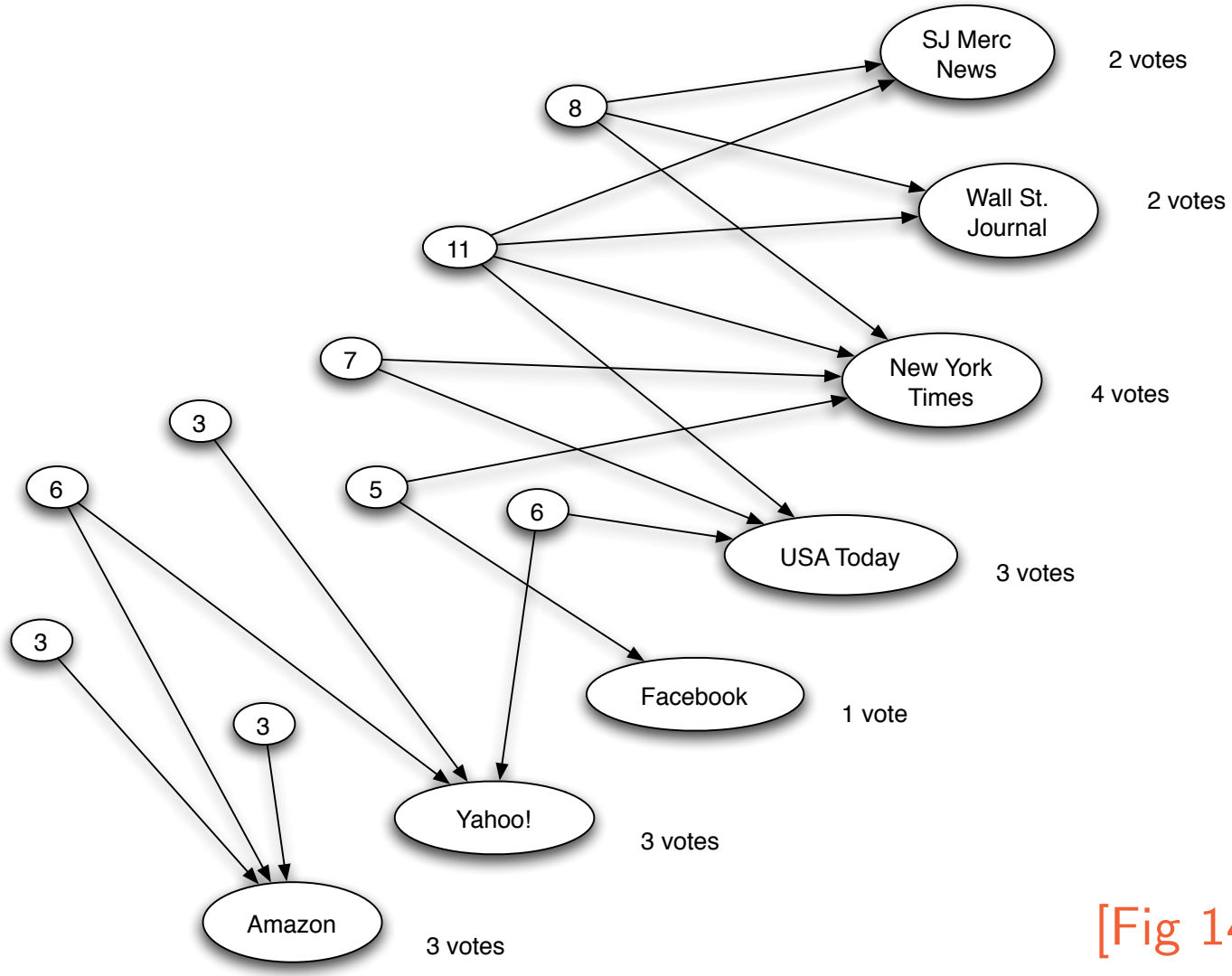
- The result of applying the authority update rule with all hub values initially 1: for each page p, auth(p) is the sum of hub values (initially just the number) of hubs pointing to p.



[Fig 14.1, E&K]

Figure: Counting in-links to pages for the query "newspapers."

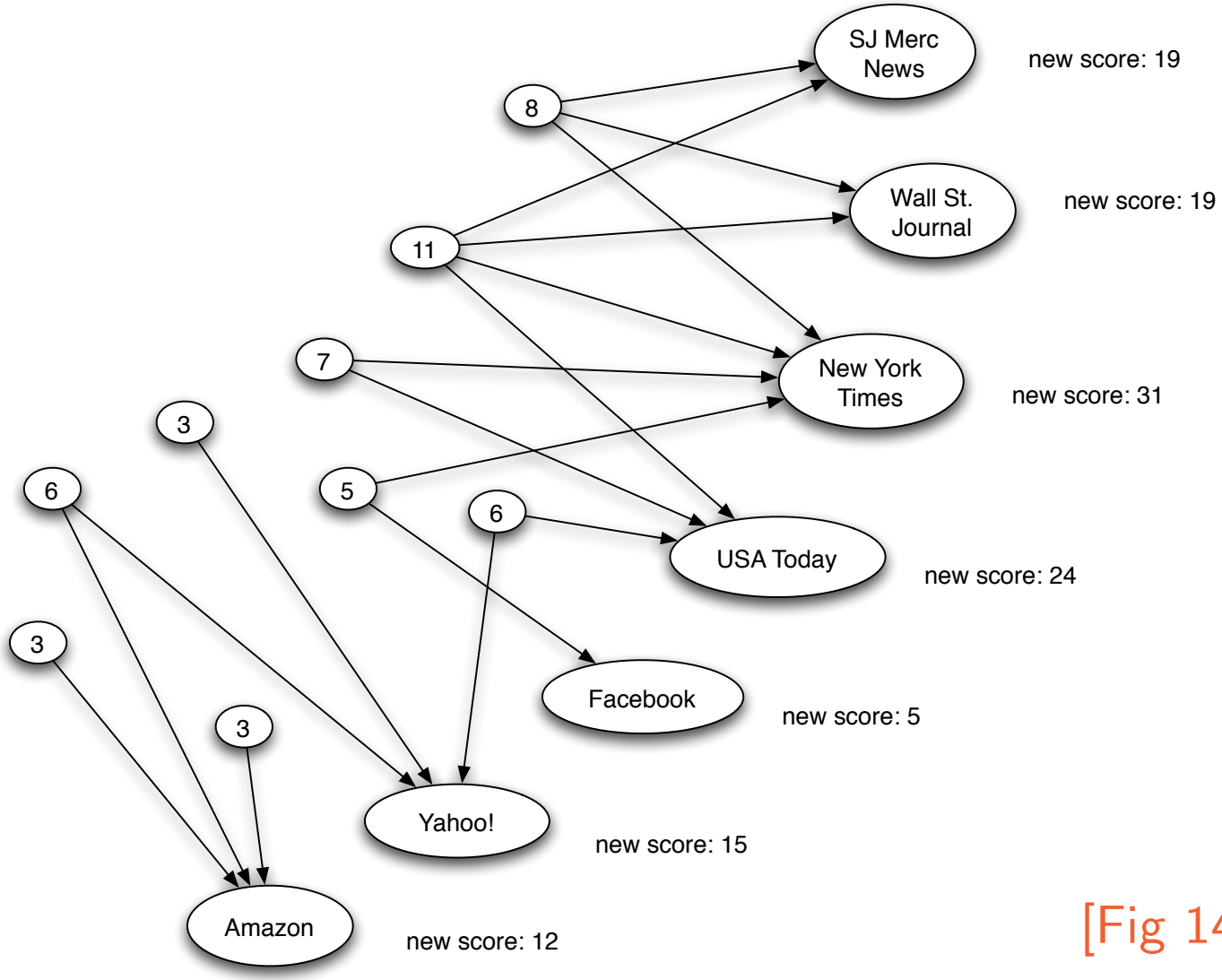
- Then to recalibrate hub values, we use the hub update rule: for each page p, hub(p) is the sum of values of all authorities that p points to.



[Fig 14.2, E&K]

Figure: Finding good lists for the query "newspapers": each page's value as a list is written as a number inside it.

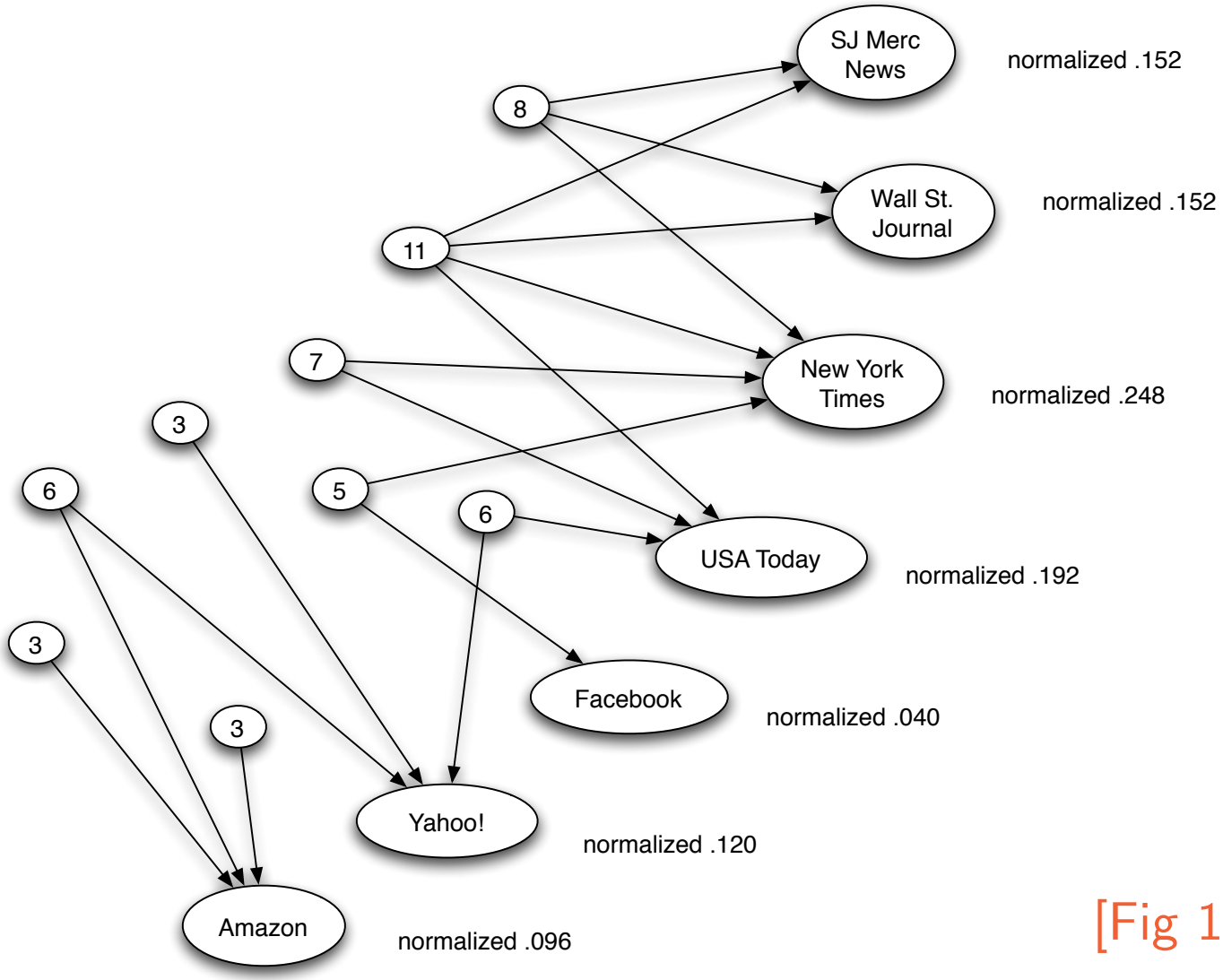
- Applying the authority update rule again we get figure 14.3.



[Fig 14.3, E&K]

Figure: Re-weighting votes for the query "newspapers": each of the labeled page's new score is equal to the sum of the values of all lists that point to it.

- Since we only care about the relative values of these numbers, both authority and hub scores can be normalized to sum to 1 (to allow convergence and avoid dealing with large numbers).



[Fig 14.4, E&K]

Figure: Re-weighting votes after normalizing for the query "newspapers".

Keep repeating a good idea

- Now having recalibrated and normalized both the authority and hub scores, we can continue this process to continue to refine these scores.

- That is, the hubs and authorities procedure is as follows:
  - Initialize all hub values to some positive vector (perhaps depending on usage or content)
  - For sufficiently large k, perform the following k times
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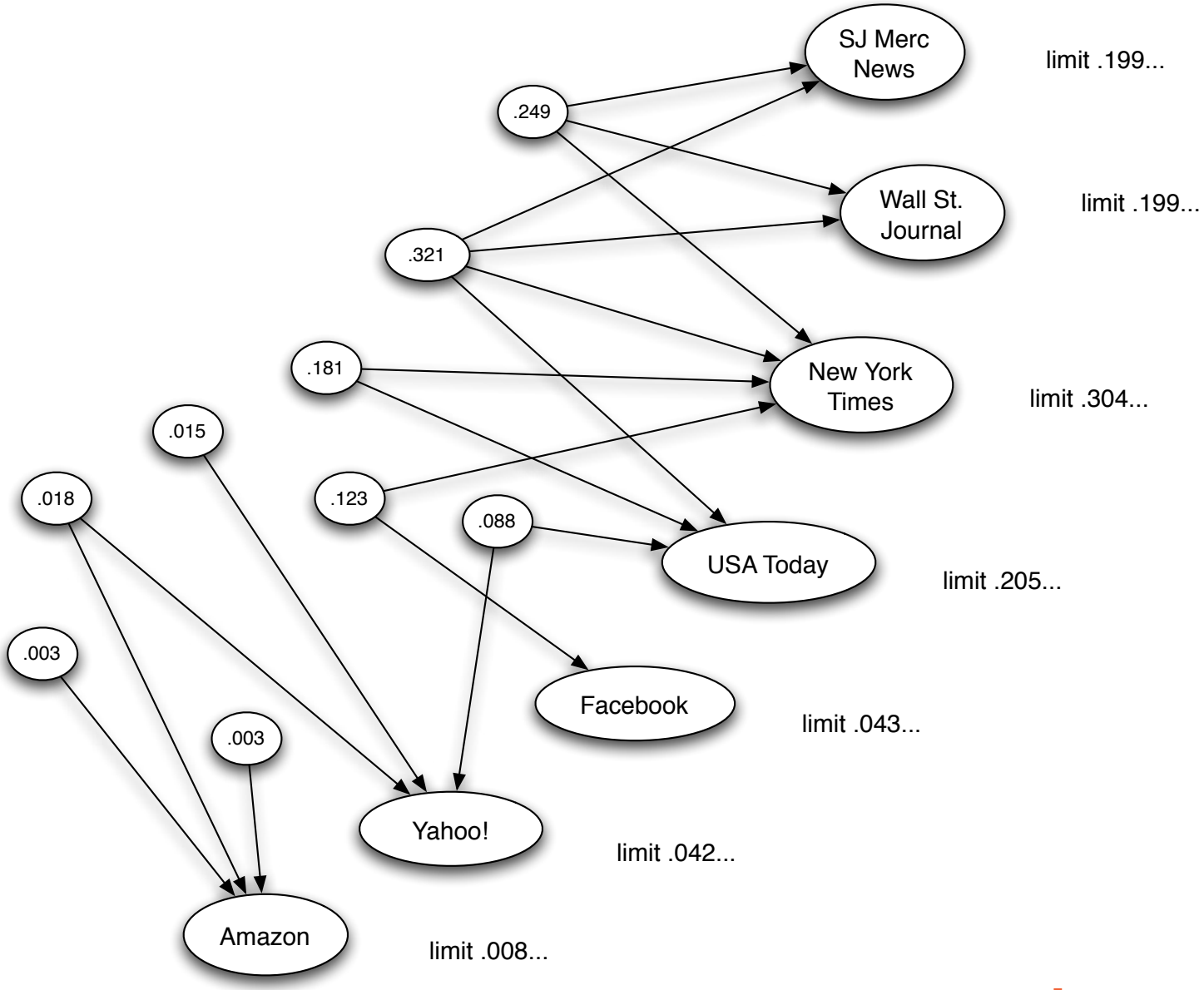
- Hubs and Authorities can be extended to work for weighted edges (e,g. weighting links in anchor text, or near a section heading, etc.)



[Fig 14.5, E&K]

Figure: Limiting hub and authority values for the query "newspapers".

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  1. Authorities directly conveying authority (without hubs)
  2. Authority values resulting from long term behaviour of a random walk on a graph

How does Page rank spread authority?

- Suppose at any point of time we have relevant authority scores
  - A page spreads its authority equally amongst all of its out links
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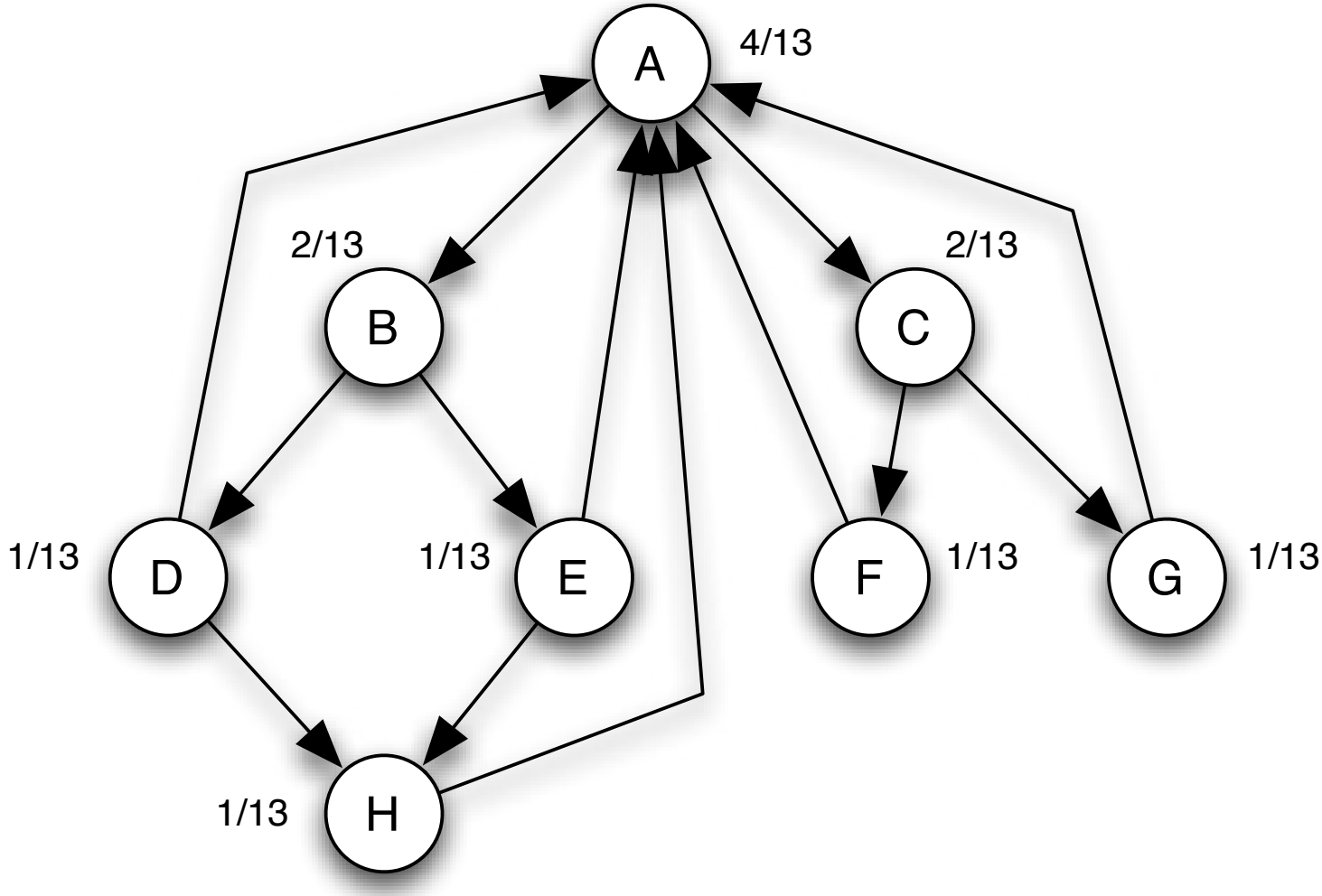
Remark

In many cases this won't reflect the desired authority. Namely, if the network has any sinks which it will surely have, then all of the authority will pass to such sinks.

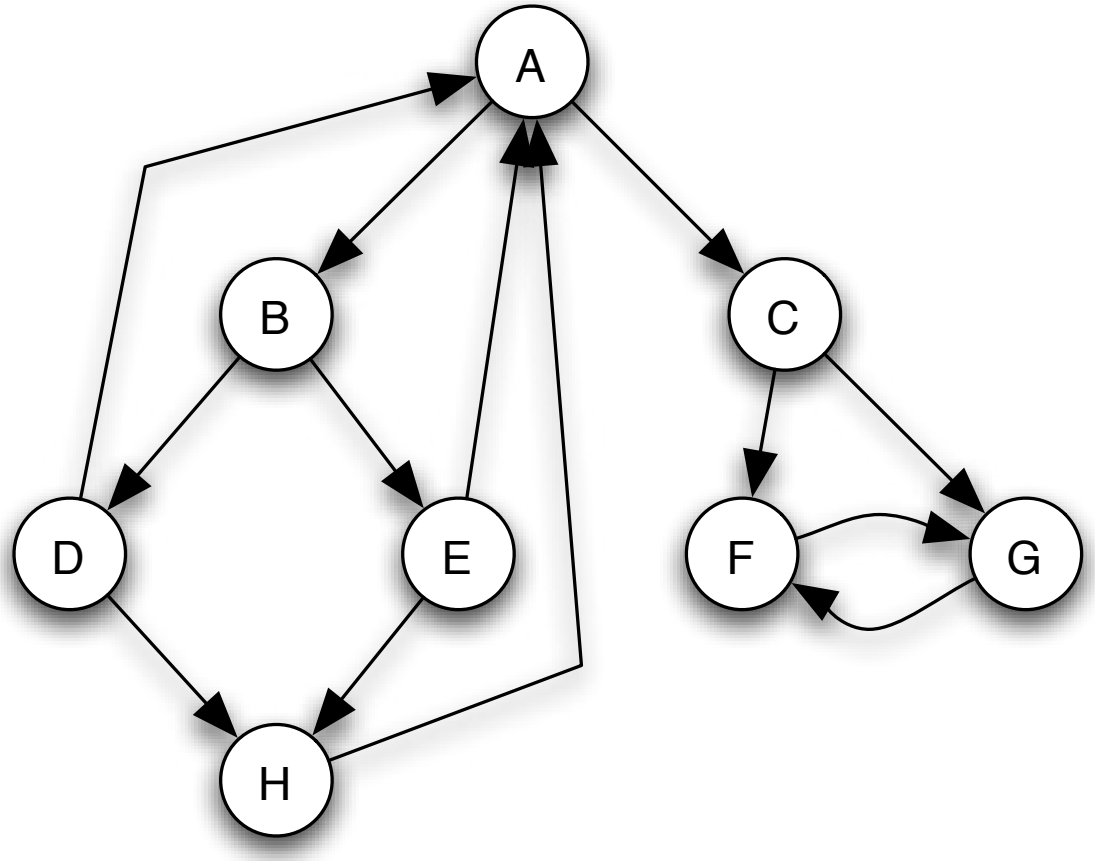
Page rank equilibrium for a network



[Fig 14.7, E&K]

Figure: Equilibrium PageRank values for the network of eight Web page.

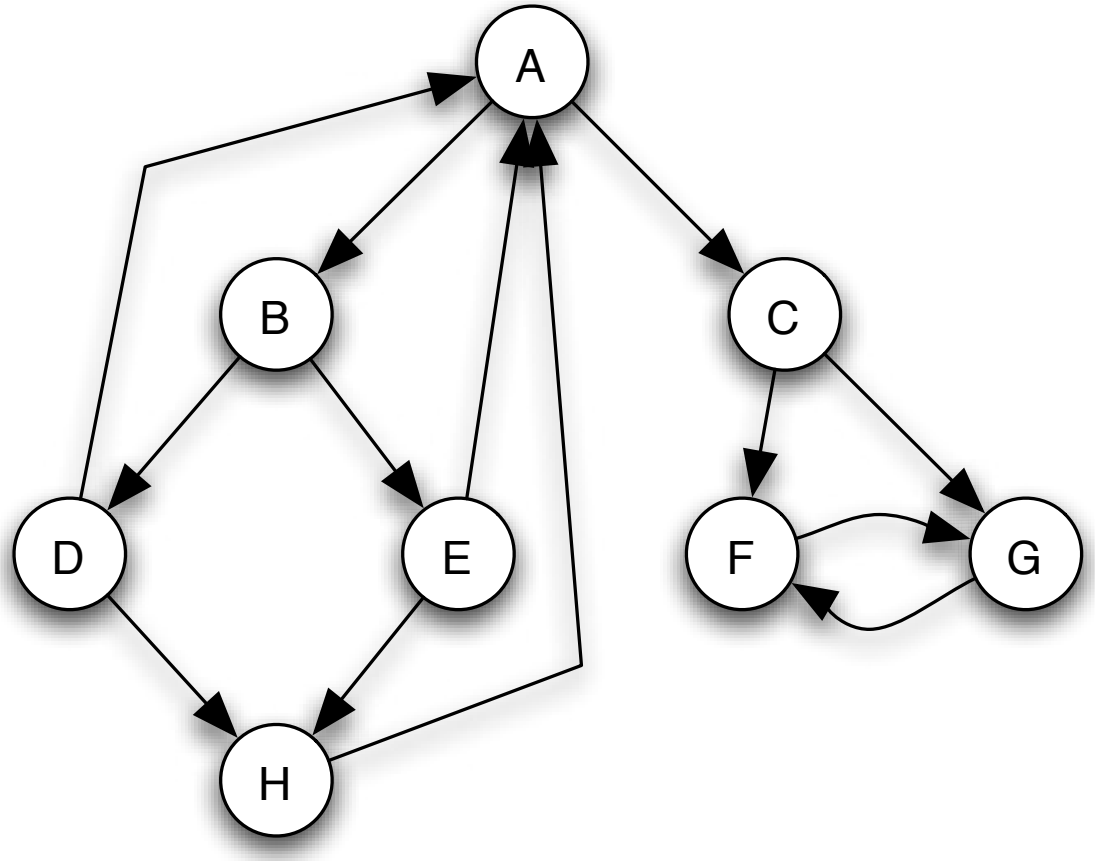
Where has all the authority gone when we redirect (F, A) and (G, A) edges?



[Fig 14.8, E&K]

The same collection of eight pages, but F and G have changed their links to point to each other instead of to A.

Where has all the authority gone when we redirect (F, A) and (G, A) edges?



[Fig 14.8, E&K]

The same collection of eight pages, but F and G have changed their links to point to each other instead of to A. Without "scaling", all the PageRank would go to F and G.

How does PageRank spread authority?

- If the network is strongly connected then there is a unique equilibrium, however it may be undesirable

Definition (Sink)

Strict (typically small) subset of the nodes with no outgoing edges that are reachable from all nodes in the network

Remark

In many cases PageRank won't reflect the desired authority. Namely, if the network has any sinks which it will surely have, then all of the authority will pass to such sinks.

Scaled page rank

- The way around this sink hole of authority is to have a scaled version of page rank where
  - only a fraction s of the authority of a page is distributed to its out links
  - the remaining $(1 - s)$ fraction is distributed equally amongst all relevant pages

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- (See the footnote on page 410 of E&K as to why in the previous example, nodes F and G will still get most of the authority but that for realistically large networks, the process works well.
  Hint: "bow-tie" structure)

Some additional remarks

- The limiting scores for both the authority and hubs approach and the page rank approach are equilibrium points for an appropriate algebraic process
- That is, if we actually were in the limiting state, we would be in the equilibrium state. In practice, we can stop the process when the change in each iteration is sufficiently small

Some additional remarks

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- That is, if we actually were in the limiting state, we would be in the equilibrium state. In practice, we can stop the process when the change in each iteration is sufficiently small
- We can weight the network edges (say according to some concept of link importance) and apply the same authority and hubs or page rank approach distributing authority in proportion to these weights

Mon. Feb 27: Announcements and Corrections

- Welcome back from reading week! :)
  - I hope it was restful
  - My sympathies to those those working on A1 using the extension – I'll see if I can adjust dates next year
- Anon feedback: Could the TAs use a digital whiteboard so it's recorded?
  - I 100% agree, unfortunately it's a hardware problem I couldn't solve in time :(
- Reminder: Critical review groups & papers due this Friday
- At my request, Professor Pitt is joining us today!
  - Today he'll be guiding you through an exercise to help us improve the course :)

Mon. Feb 27: Announcements and Corrections

M1-999-1

- Midterm on March 10th
  - In tutorial
  - All material covered before reading week (i.e., before this slide)
  - 50 minutes, 5 questions
  - Handwritten aid-sheet allowed (letter size)
  - First page of midterm, and a practice midterm, are on the course website
    - ★ See "Tests" tab

CSC303 Midterm, Winter 2023

March 10th, 1:10-2:00PM

Before the test begins, please read the front page carefully and fill out your name and student number below. Please do not turn the page until the test begins.

Student Number: _____

Given Name(s): _____

Family Name(s): _____

The only resource you are allowed to consult is a handwritten aid sheet, letter size (8.5 × 11 inches), double-sided.

If you have a question then please raise your hand; the TA invigilator will acknowledge and fetch the instructor to answer your question.

You will receive 20% of the points for any (sub)problem for which you write "I do not know" You will receive 0% if you leave a question blank. If instead you submit irrelevant or erroneous answers, you will receive 0 points. You may receive partial credit if you explain why you can't answer a question, and the explanation shows understanding of relevant course material. You will receive partial credit for the work that is clearly "on the right track."

This test is 8 pages, and 5 questions; when the test begins, please check that they are all present.

Please do not write on the back of pages, as those will not be scanned! If you need more space, then pages 2 and 8 are blank and will be scanned. Note that all the questions have fairly short answers, so you shouldn't need to use page 2 or 8.

Question	Out Of
Q1	15
Q2	10
Q3	7.5
Q4	10
Q5	12.5
Total	55

Remember, you are far more than just a grade, and you have already shown yourself to be a wonderful, talented, and intelligent human being just by making it this far. Whatever happens can never change that. Good luck! :-)

Advanced material (section 14.6): Handwaving argument why these processes converge

Both the page rank and hubs and authorities processes can be understood as a linear transformation

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Both the page rank and hubs and authorities processes can be understood as a linear transformation

- Consider a web network of n pages
  - We can represent the hub, authority or page rank values at any time k of the process by an n-dimensional (column) vector, denoted (respectively) by $\mathbf{h}^{(k)}, \mathbf{a}^{(k)}, \mathbf{r}^{(k)}$.

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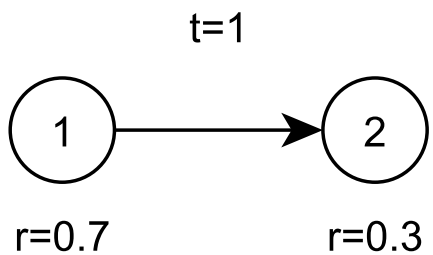

$$\mathbf{r}^{(1)} =< 0.7, 0.3 >$$

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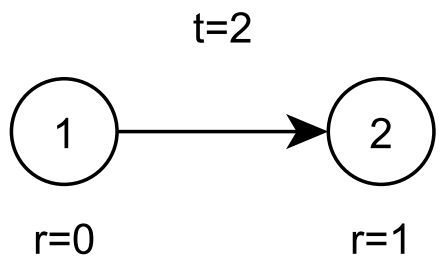

$$\mathbf{r}^{(2)} =< 0, 1 >$$

Advanced material (section 14.6): Handwaving argument why these processes converge

Let $\mathbf{v}$ be any of the hub, authority or page rank vectors

- PageRank & Hubs and Authorities can both be viewed as a linear transformation $\mathbf{v}^{(k+1)} = M\mathbf{v}^{(k)}$
- M is an appropriate $n \times n$ matrix, whose entries are non negative real numbers
  - Why $n \times n$?

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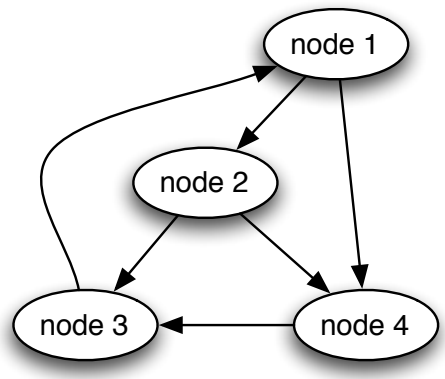
- M is an appropriate $n \times n$ matrix, whose entries are non negative real numbers
  - Why $n \times n$? We have n webpages, each has a value before & after the update
- Furthermore, there are conditions that will guarantee the convergence of the process
  - i.e., when there exists $\mathbf{v}^{(*)} = \lim_{k\to\infty} \mathbf{v}^{(k)}$ and when this limit vector $\mathbf{v}^{(*)}$ is unique and independent of the starting vector $\mathbf{v}^{(0)}$

Advanced material continued: page rank convergence

- Figure 14.3 of the text illustrates a simple directed graph and the matrix N that defines the unscaled page rank update process. That is, $\mathbf{r}^{(\mathbf{k}+\mathbf{1})} = N^T \mathbf{r}^{(\mathbf{k})}$ where N^T is the transpose of matrix N.



$$\begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

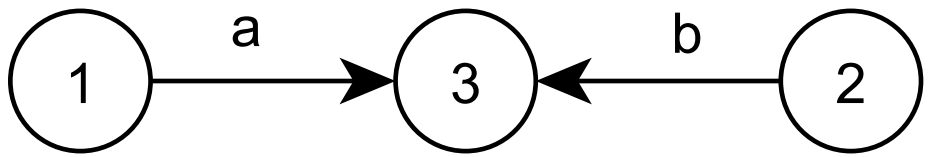
[Fig 14.13, E&K]

Figure: A toy web graph and the associated matrix N describing the unscaled update process.

- N is a weighted adjacency matrix
  - Weights are the proportion of authority that's transferred along the edge

Advanced material continued: page rank convergence

- Note: N_{ij} is the proportion of node i's rank, that should go to node j in the next update
  - Therefore, N_{ij}^T is the proportion of j's rank that i should receive



- Let's focus on node 3
- We can see that $r_3^{k+1} = ar_1^k + br_2^k + r_3^k$

$$N = \begin{bmatrix} * & * & a \\ * & * & b \\ * & * & 1 \end{bmatrix}$$

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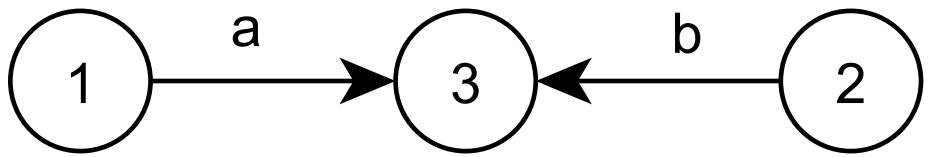

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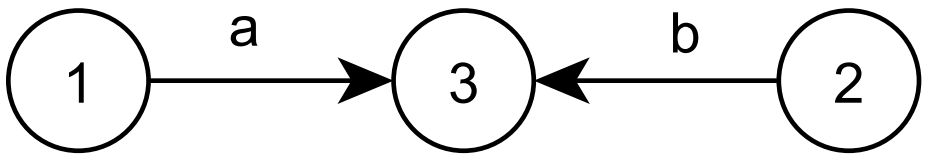

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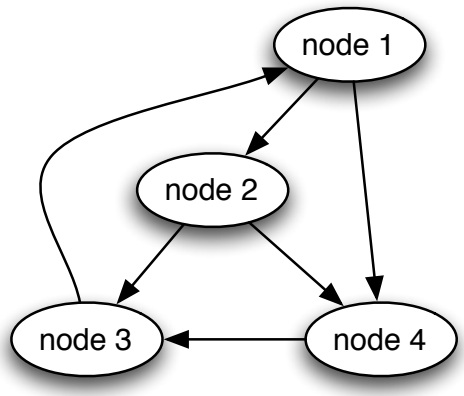
Page rank analysis for the scaled update

Similarly Figure 14.4 illustrates the same graph and the matrix $\tilde{N}$ that defines the scaled page rank update process with scaling factor $s = .8$.



$$\begin{bmatrix} .05 & .45 & .05 & .45 \\ .05 & .05 & .45 & .45 \\ .85 & .05 & .05 & .05 \\ .05 & .05 & .85 & .05 \end{bmatrix}$$

[Fig 14.14, E&K]

Figure: The same toy web graph and the associated matrix $\tilde{N}$ describing the scaled update process with $s = 0.8$.

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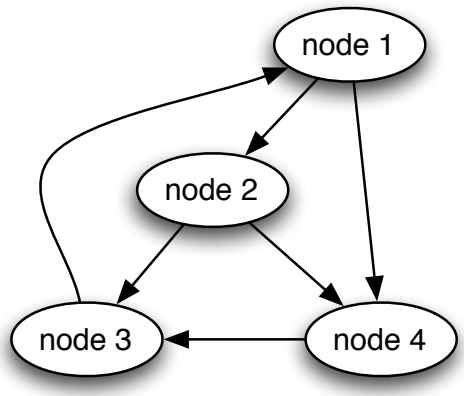

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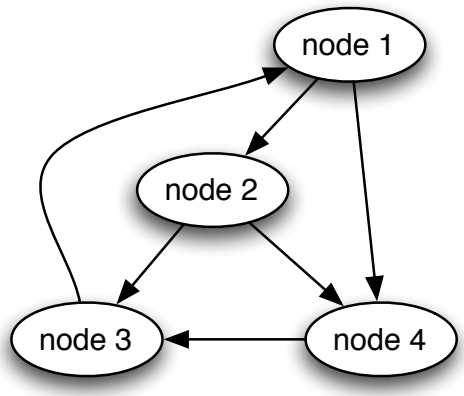

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- $\tilde{N}^T = sN^T + (1-s)\frac{\mathbf{1}(\mathbf{1}^T)}{n}$
- It follows that $\mathbf{r^{(k)}} = (\tilde{N}^T)^k \mathbf{r^{(0)}}$
- If the process is converging then it would be converging to some $\mathbf{r}^*$ satisfying $\mathbf{r}^* = \tilde{N}^T \mathbf{r}^*$

Now comes the necessary linear algebra

Matrices are more than a convenient representation of the process; linear algebra review time!

- Let $M_{n \times n}$ be a full rank matrix. Recall that the matrix-vector multiplication $M\mathbf{v}$ can rotate and expand/shrink the vector $\mathbf{v}$
  - Simpler to visualize the meaning of such a linear transformation in 2-space or 3-space

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More linear algebra

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- Since $\lambda_1 > |\lambda_i|$ for $i > 1$, and since every vector is a linear combination of the eigenvectors, it follows that as $k \to \infty$, the transformation M^k is being dominated by the largest eigenvalue acting on its associated eigenvector
- For the scaled matrix $\tilde{N}^T$, all entries are positive and the largest eigenvalue is 1
  - Therefore as $k \to \infty$, $(\tilde{N}^T)^k \mathbf{v}$ will converge to the eigenvector $\mathbf{y}$ associated with the largest eigenvalue 1

Aside: Random Walk

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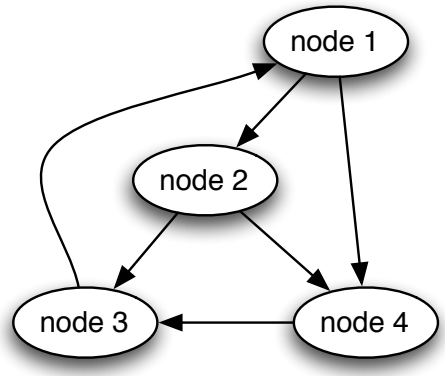
  - We can view this as a Markov chain
  - i.e., a type of probabilistic finite state machine that's represented as a graph, where each timestep we follow an edge based on the corresponding probabilities
- Similarly, scaled page rank can be viewed as the same Markov chain but with added low probability edges between every pair of nodes

Aside: Random Walk



$$\begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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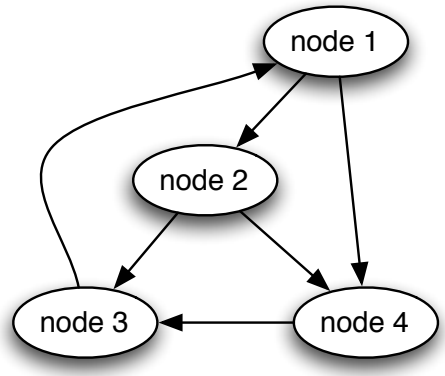

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- Under the assumption that the chain is *irreducible* (we can reach state j from all states i and vice versa), and *aperiodic* (there is no state i such that if you leave i, you can only return on timesteps that are multiples of some $p > 1$) then there is a unique *stationary distribution* π that the chain converges to
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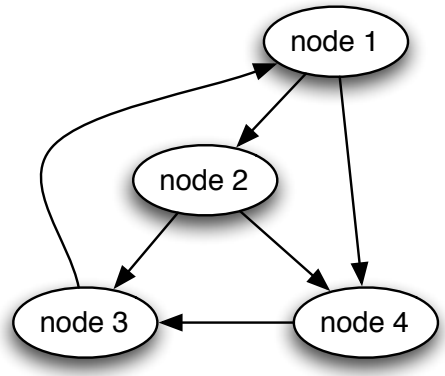

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- This is the distribution that we found
- When π exists, then all initial probability distributions over the nodes converges to π

Similar analysis for hubs and authorities

- If M is the adjacency matrix of the web graph, then the (unnormalized) process can be described by $\mathbf{h} = M\mathbf{a}$ and $\mathbf{a} = M^T\mathbf{h}$.

- Exercise: Convince yourself this is true!

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- Exercise: Convince yourself this is true!
- Then
  1. $\mathbf{a}^{(1)} = M^T\mathbf{h}^{(0)}$
  2. $\mathbf{h}^{(1)} = M\mathbf{a}^{(1)} = MM^T\mathbf{h}^{(0)}$
- It follows that
  1. $\mathbf{a}^{(k)} = (M^TM)^{k-1}M^T\mathbf{h}^{(0)}$
  2. $\mathbf{h}^{(k)} = (MM^T)^k\mathbf{h}^{(0)}$

Hubs and authorities analysis continued

- The matrices (MM^T) and (M^TM) are symmetric and have non-negative entries

- Any $n \times n$ symmetric matrix S with non negative entries has an orthonormal set of n eigenvectors all of whose associated eigenvalues are real
  - By normalizing the scores, we assume that the largest eigenvalue $\lambda_1 = 1$

Hubs and authorities analysis continued

- The matrices (MM^T) and (M^TM) are symmetric and have non-negative entries
- Any $n \times n$ symmetric matrix S with non negative entries has an orthonormal set of n eigenvectors all of whose associated eigenvalues are real
  - By normalizing the scores, we assume that the largest eigenvalue $\lambda_1 = 1$
- If the largest eigenvalue is unique (which is what would happen in a real web graph), then the same analysis for page rank applies (assuming that the starting hub scores are all positive).

Returning to the issue of influence

In some sense or another we are often talking about social influence in this course. Even in web ranking (Ch 14), we can view hubs as influencing which Web pages will be ranked highly.

In chapter 18, we observed two sequential processes where previous individual decisions had a significant impact:
1) The evolution of links on the Web, and
2) The evolution of opinions in evaluating music.

The music evaluation experiment is closer to reality in the sense that it explicitly integrates a measure of quality (a simplification of selection?) into the decision making process.

Recap

- Power Laws
  - Definition
  - Zipf's law
  - Dynamics that often give rise to power laws
- Rich get Richer model
- Salganik et al. music popularity experiment
- Ranking web results
  - Hubs and Authorities
  - (Scaled) Page Rank
  - Proof-sketches of convergence