Social and Information Networks

University of Toronto CSC303
Winter/Spring 2023

Week 4: Jan 30-Feb 3

This week's agenda

Last week (Chapter 4 of the text):

- Schelling's segregation model
- triadic, focal, and membership closure
- the probability of a closure as a function of the number of common friends, common interests (foci), or friends in a given focus

This week:

- Chapter 5 and structural balance

This week's agenda

- Structural Balance
  - Balanced triangles
  - Strongly balanced networks
  - Strong balance theorem
  - Weak structural balance
  - The signed Laplacian matrix

Structural balance: positive and negative links

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- From assuming strong & weak edges, we were able to infer properties of social networks
- Can we do something similar with positive vs. negative edges?
  - Can local properties (e.g., how edges of a triangle are labeled) can have global implications?
  - Are there any provable results about network structure?

Some initial assumptions

We start with a strong assumption:
Assume the network is a complete (undirected) graph. That is, as individuals we either like or dislike someone. Furthermore, this is not nuanced in the sense that there is no differentiation as to the extent of attraction/repulsion).

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[Image modified from *Star Wars: Episode III - Revenge of the Sith*. Directed by George Lucas, Lucasfilm, 2005]

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Aside: The text also reflects a little on the nature of directed networks (when discussing the *weak balance property*) but essentially this chapter is about undirected networks.

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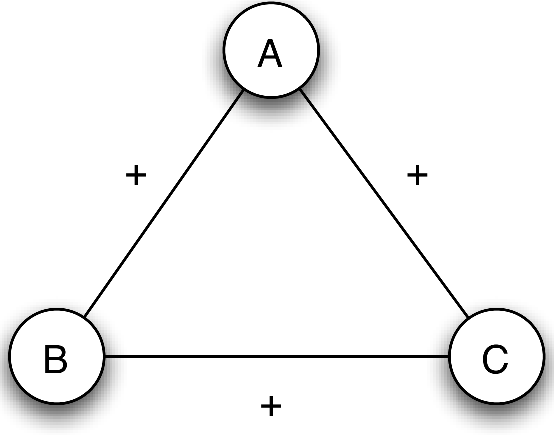
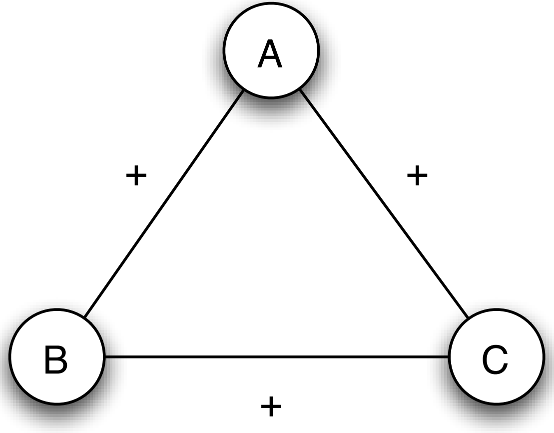
Using a central idea from social psychology, some of the four triangle labellings are considered relatively stable (called *balanced*) and the rest are relatively unstable (*not balanced*).
Here follows the four types of triangles as depicted in Figure 5.1 of the text:


A
+
+
B
C
+

A natural stable configuration



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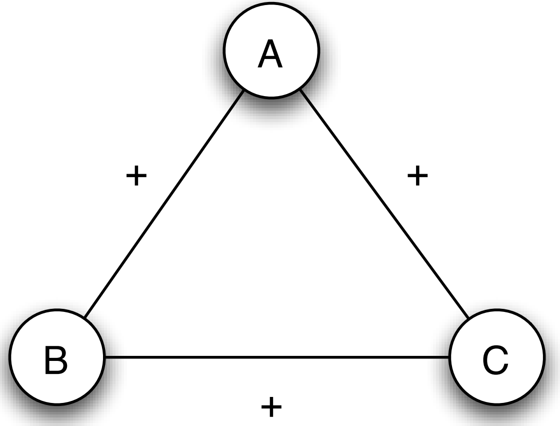
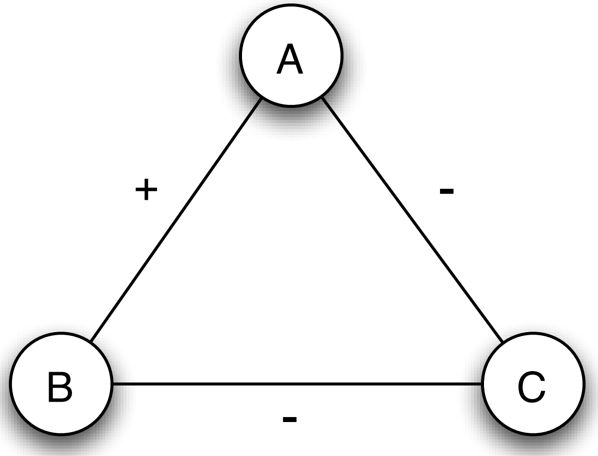


In this case, A, B, C are mutual friends and that naturally indicates that they would likely remain so.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+
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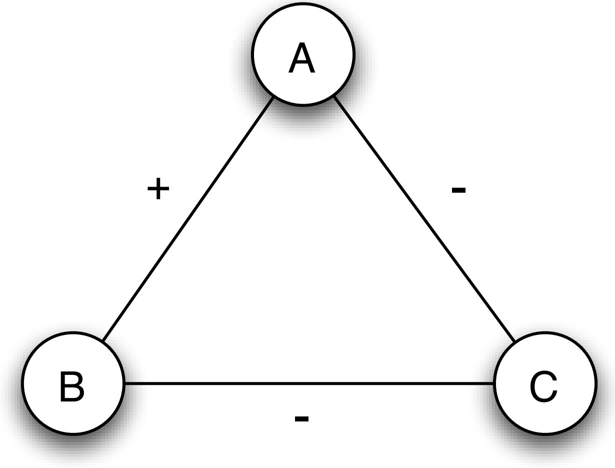
The second stable configuration



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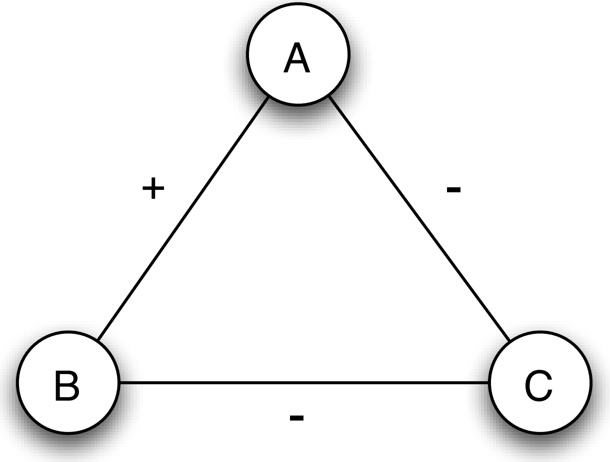
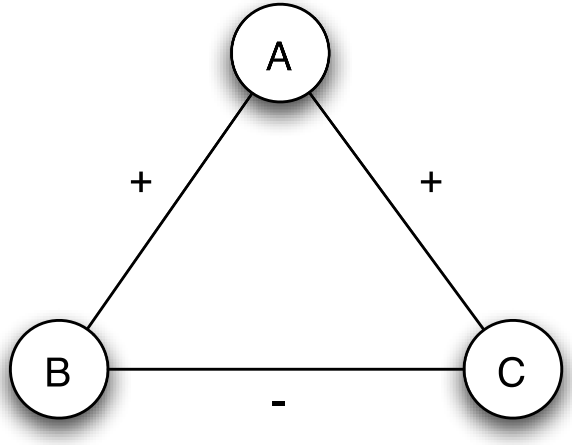


This may be a slightly less obvious stable situation where A and B are friends and if anything that friendship is reinforced by a mutual dislike for C.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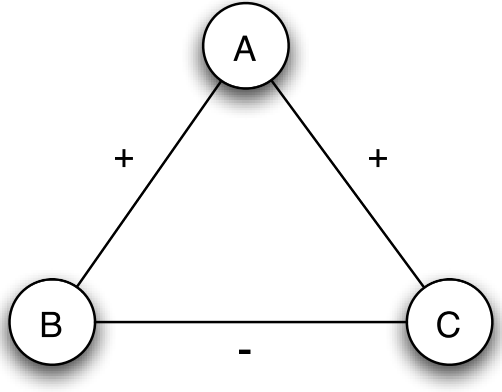

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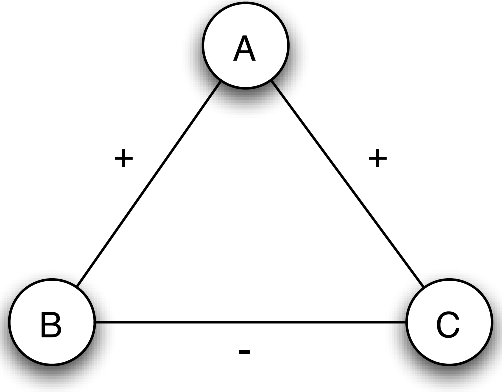

- A has two friends B and C who unfortunately do not like each other
- Look familiar?

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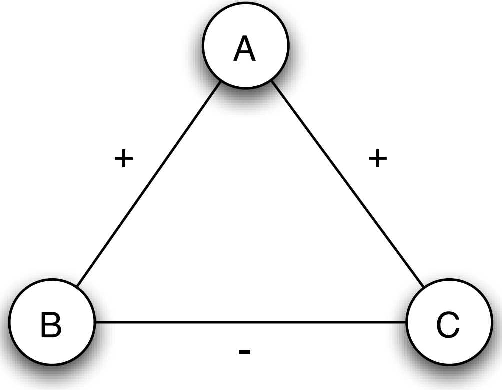

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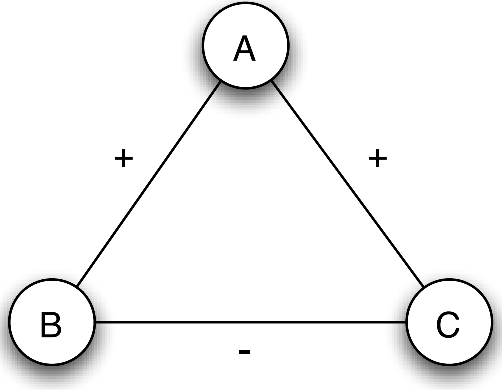
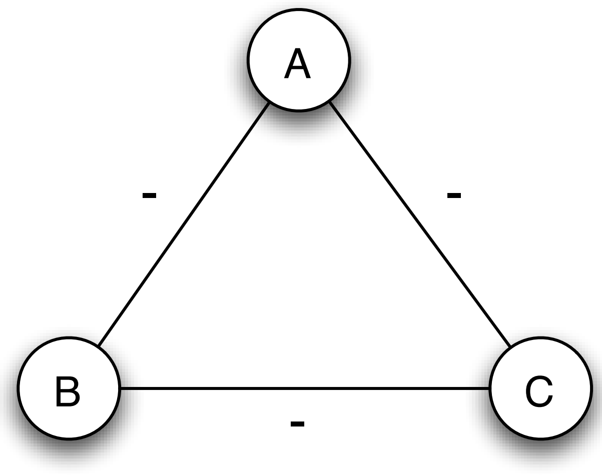


- *A* has two friends *B* and *C* who unfortunately do not like each other
- Look familiar? Latent stress!
- Claim: the stress of this situation will encourage *A* to either make *B* and *C* become friends, or for *A* to take a sides with either *B* or *C*, thus moving toward the previous stable configuration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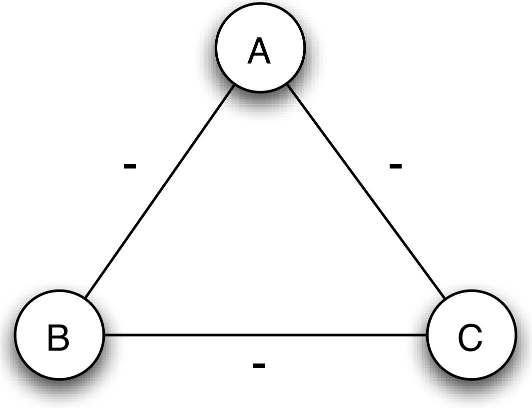

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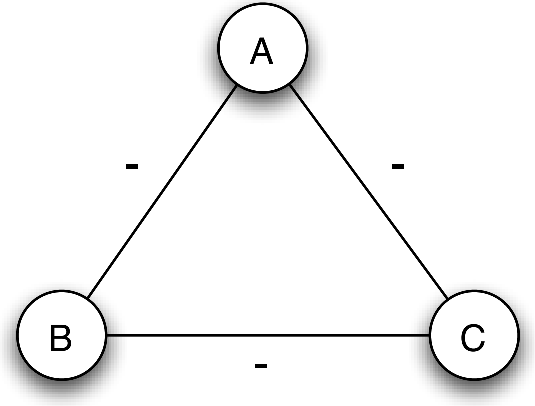

Why is this called unstable?

- "the enemy of my enemy becomes my friend", as sometimes seen in international relations
- This particular triangle has some nuances, we'll revisit it soon

The strong structural balance property

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In order to try to understand if this theory tells us anything about the global structure of the network, we can make the following strong balance assumption (much as we made the strong triadic closure assumption).

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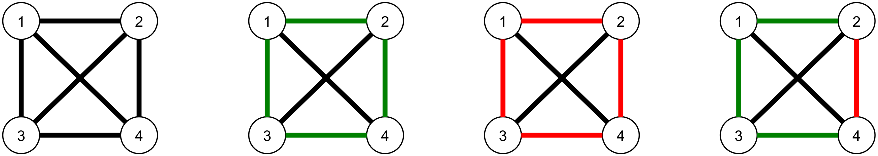
- Therefore, strong structural balance constrains all n choose 3 triangles
- Like strong triadic closure, this is clearly an extreme & unrealistic assumption
- However, like STC we hope this strong assumption will also suggest useful information about the network

Demonstration time! Do we have 4 volunteers from the audience?



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Later in the term, we will discuss stable matchings. (How many have seen this in CSC304 or elsewhere?) We view stable matchings as an equilibrium. In stable matchings (as in balanced triangles), it is a pair of "agents" that we consider in a single change. We discuss stable matchings later in this course.

Consequence of the strong structural balance property: A provable characterization of networks that satisfy the property

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We can also satisfy the property with two communities such that all intracommunity edges are friendly, and all cross-community edges are negative

- Imagine we had two communities of active political people (e.g. $X =$ the "base" for candidate or political party R, and Y and the "base" for candidate or political party B
- In the world of highly politicized politics, it isn't too far of a stretch to think that everyone within a community are friends and everyone dislikes people in the other community

Consequence of the strong structural balance property: A provable characterization of networks that satisfy the property

So far: two possibilities

1. the network is a clique with all positive edges
2. the network is composed of two positive cliques with a complete bipartite graph of negative edges between the communities

Are there other possible ways to have the strong balance property?

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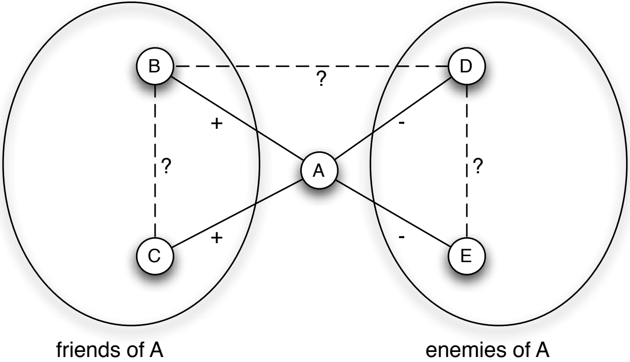
Proof
We assume that the network satisfies the strong balance property. If there are no enemies, then we are done. So suppose there is at least one negative edge and for definiteness lets say that edge is adjacent to node A. Let X be all the friends of A and Y all of its enemies. So every node is in either X or Y since every edge is labelled.

Proof of balance theorem continued

Consider the three possible triangles as in the figure. It is easy to see that in order to maintain structural balance, B and C must be friends as must D and E, whereas B and D (also C and E) must be enemies.



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Aside: Of course, this immediately raises the question as to how many existing edge labels need to be changed so that a complete network is balanced (or an incomplete network can be made to be balanced)? And will networks tend to dynamically evolve into balanced networks. But for now we will assume that all existing labels are permanent.

How to label missing edges?

When considering the strong triadic property, if all existing triangles satisfied the strong triadic property, then there was always a trivial way to assign labels to unlabelled edges by simply making each unlabelled edge a weak link.

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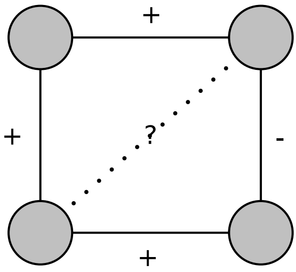

How to label missing edges?

It is easy to see that this is not always possible. For example, consider a network which is a 4 node cycle having 3 positive edges and one negative edge. Any way to label a "diagonal edge" will lead to an imbalance.

We are then led to the following
Question: Can we determine when there is an efficient algorithm to complete the network so as to satisfy the strong balance property? And if there is a completion, how efficiently can one be found?

Determining when and how to complete a network to satisfy the strong balance property

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The previous example of a 4 node cycle is a clue as to how to proceed. That example can be stated as follows: if a network contains a 4 node cycle with one negative edge then it cannot be completed (to be strongly balanced) . More generally, if a network contains a cycle (of any length) with one negative edge, it cannot be completed. And even more generally, if a network contains a cycle having an odd number of negative edges it cannot be completed. Why?

Consequence of an odd cycle

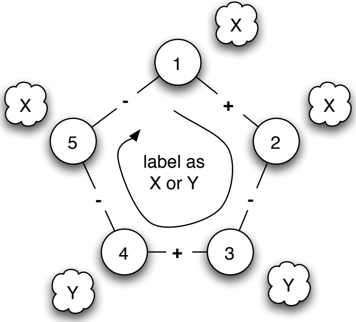

Figure 5.10: If a signed graph contains a cycle with an odd number of negative edges, then it is not balanced. Indeed, if we pick one of the nodes and try to place it in X, then following the set of friend/enemy relations around the cycle will produce a conflict by the time we get to the starting node.

The algorithm for determining if a partially labelled network can be completed to the strongly balanced

Lets call a cycle with an odd number of negative edges an odd cycle. The desired algorithm will either find an odd cycle (certifying that the network cannot be completed) or it will return a bipartiton of the nodes satisfying the Balance Theorem. This then also determines if a complete network is balanced.

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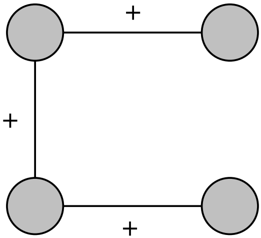
We proceed as follows:

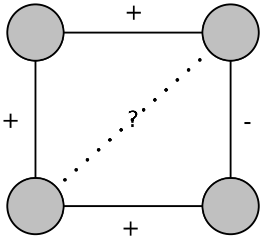
- Suppose $G = (V, E)$ is the given connected network and let $G^+ = (V, E^+)$ where $E^+ = \{e \in E$ such that e is a positive link.$\}$
- We consider the connected components $\mathcal{C} = C_1, \ldots, C_r$ of G^+.
- Note that all edges between any C_i, C_j must be labelled as negative edges (or else they would have been merged into a larger connected component in G^+).
- For every C_i, we must check if there is a negative edge between two nodes in C_i. If so then there is a cycle in C_i with one negative edge, and hence C_i (and thus G) cannot be completed.

The algorithm for determining if a partially labelled network can be completed to the strongly balanced

Connected positive component $\mathcal{C}_i$



Negative edge produces an odd cycle



Completing the algorithm

- Otherwise, consider the graph $G^- = \{\mathcal{C}, E^-\}$ whose nodes are the components of G^+ and whose edges are negative edges in G.
- Since G is connected, G^- is connected.
- if G^- has a cycle with an odd number of negative edges, then by following positive edges in each C_i we have such a cycle in G. We then again have a witness that G cannot be completed.
- Otherwise we are showing that G^- is bipartite and this gives us the bipartition we need for the balance theorem.
- A graph has an odd cycle iff the graph is not bipartite. Breadth first search can be used to determine whether or not a graph is bipartite (equivalently has a 2-colouring). Hence this development is efficient.

We now return to the assumption that our networks are undirected complete graphs.

Recap

- Structural Balance
  - Balanced triangles
  - Strongly balanced networks
  - Strong balance theorem

Wed. Feb 1: Announcements and Corrections

- Typo on slide 28 of this week's slides has been corrected:
  - "odd cycle": An odd number of *negative* edges
  - Thank you to the eagle-eyed student who spotted the typo! :)
  - I've also clarified the MBS definition on slide 50
- Critical Review project has been released
  - See `https://www.cs.toronto.edu/~ianberlot/303s23/assignments.html`
- Today's 10PM Zoom-only office hours is moved to tomorrow, Thursday 2nd at 10PM
  - Today's in-person 2PM office is unaffected

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Comic from xkcd

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- A *draft* is due March 24th [submission details to follow]

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Critical Review

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- You have an upper limit of 5 pages, but you're free to make it long or short as you feel is appropriate (the exemplars are around 1000 words, thereabouts or a bit longer is definitely reasonable)

Friends-enemies vs trust-distrust

- There is always an ambiguity in social networks in how we interpret links
  - Is a friend as we might traditionally mean a “good friend”, or is it a friend as in Facebook friend (i.e., often an acquaintance)
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- Example time!

The ambiguity in the trust-distrust relation

One distinction is that trust may be more of a directed edge concept relative to friendship.

Ignoring the fact that trust might not be at all symmetric, there is an additional ambiguity in the trust-distrust terminology. Namely, the text considers two possible interpretations that are meaningful even in the context of a simple setting as in the online product rating site Epinions.

1. If trust is aligned with agreement on polarized political issues, then the four cases of balanced and unbalanced triangles still seem to apply. In particular, if A distrusts B and B distrusts C, it is reasonable to assume that A trusts C and hence a triangle having three negative labels is not stable.

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2. However, if A distrusts B is aligned with A believing that he/she is more knowledgeable than B about a certain product, then a triangle having three negative labels is stable.

This suggests that it is reasonable to study a weaker form of structural balance.

A weaker form of structural balance

It is then interesting to consider a weaker form of structural balance where the only unstable triangles are those having two positive labels.

Definition (Weak Structural Balance)

A network satisfies the *weak structural balance property* if it does not contain any triangles with exactly two positive edges.

Question: Is there a characterization of which (complete) networks satisfy the weak structural balance property?

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Theorem: A network $G = (V, E)$ satisfies the weak structural balance property iff $V = V_1 \cup V_2 \ldots V_r$ such that all edges within any V_i are positive edges and all edges between V_i and V_j $(i \neq j)$ are negative edges.

Proof of the characterization of weak structural balance

Clearly if the network $G = (V, E)$ has the network structure specified in the Theorem, then the network satisfies the weak balance property. The converse (that the weak balance property implies the network structure) is a reasonably simple inductive argument (say with respect to the number of nodes).

Proof of the characterization of weak structural balance

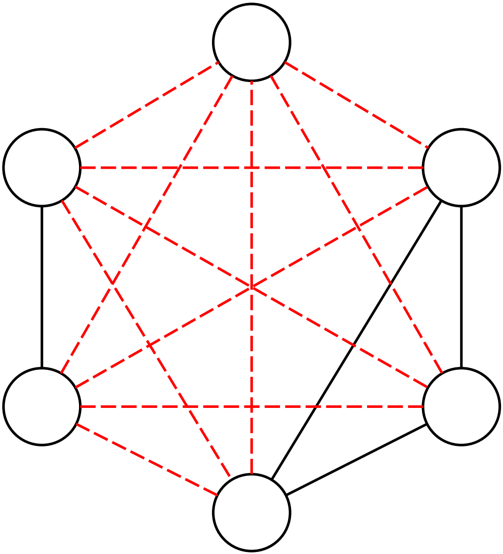
Clearly if the network $G = (V, E)$ has the network structure specified in the Theorem, then the network satisfies the weak balance property. The converse (that the weak balance property implies the network structure) is a reasonably simple inductive argument (say with respect to the number of nodes).

Consider any node A and let X be all the friends of A.
The following two claims are easy to verify:

- Any $B, C \in X$ are friends
- If $B \in X$ and $D \notin X$, then B and D are enemies.

Upon removing the nodes in X, the induced network G' of the remaining nodes still must satisfy the weak structure balance property and hence by the induction hypothesis must have the stated network structure.

Example: Partitioning a weakly balanced graph



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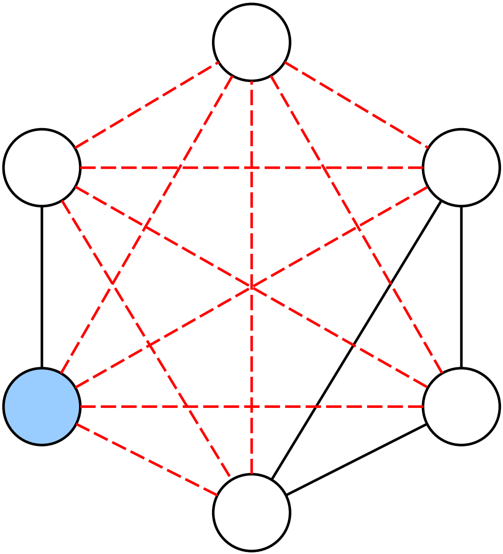

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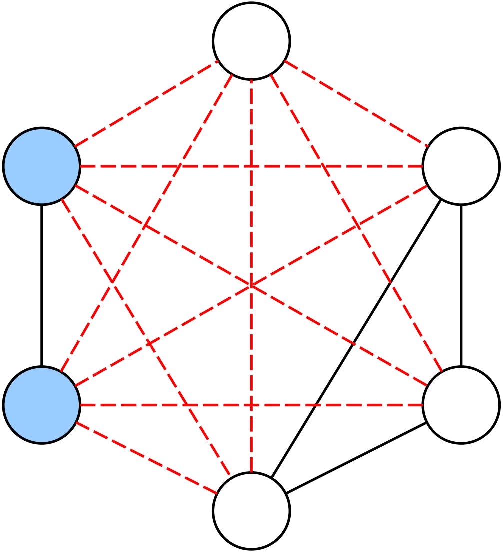

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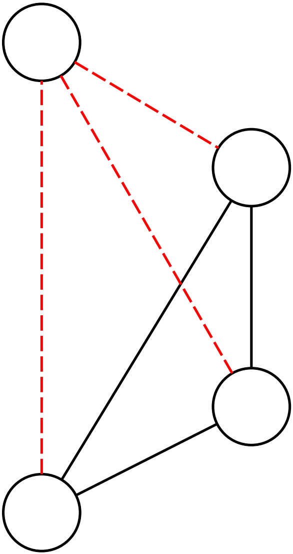

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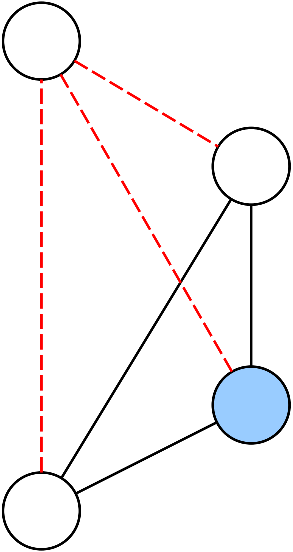

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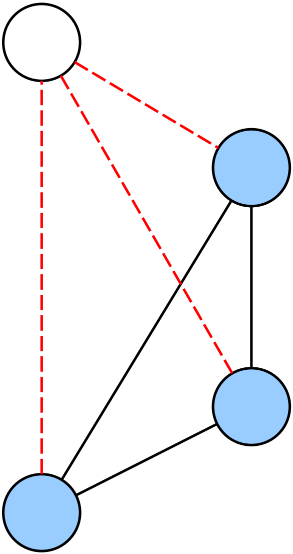

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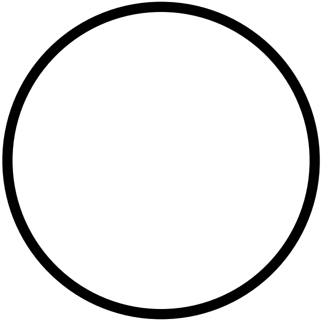

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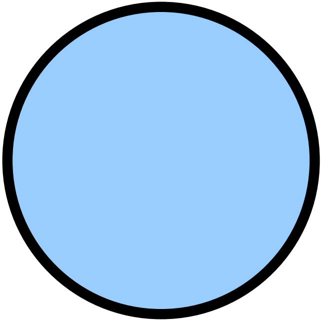

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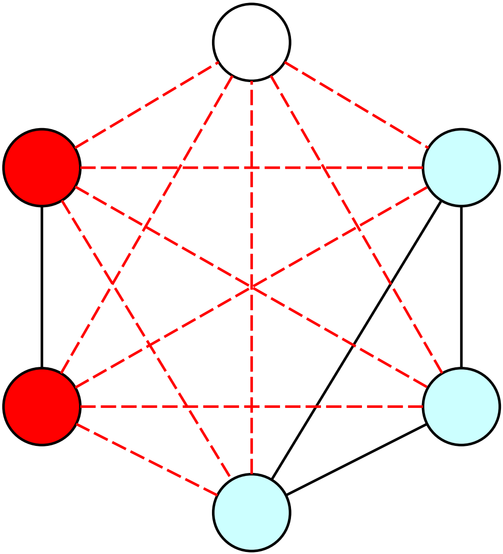
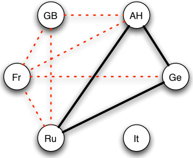
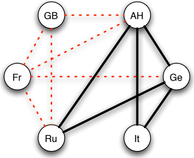


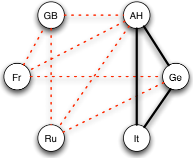
The evolution of European alliances preceding WWI



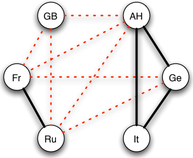
(a) *Three Emperors' League 1872–81*



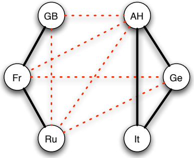
(b) *Triple Alliance 1882*



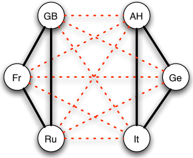
(c) *German-Russian Lapse 1890*



(d) *French-Russian Alliance 1891–94*



(e) *Entente Cordiale 1904*



(f) *British Russian Alliance 1907*

Figure 5.5: The evolution of alliances in Europe, 1872-1907 (the nations GB, Fr, Ru, It, Ge, and AH are Great Britain, France, Russia, Italy, Germany, and Austria-Hungary respectively). Solid dark edges indicate friendship while dotted red edges indicate enmity. Note how the network slides into a balanced labeling — and into World War I. This figure and example are from Antal, Krapivsky, and Redner [20].

Efficiently finding balanced subgraphs

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Definition (MBS)

Given a signed graph $G = (V, E, w)$, MBS is the problem of finding the *maximum balanced subgraph*. i.e. finding the largest $V' \subseteq V$ such that $G' = (V', \{(v_1, v_2) \in E | v_1, v_2 \in V'\}, w)$ is strongly balanced (or completable to such).

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- Problem is NP-Hard, so we have to approximate
- We're going to do this, by studying the properties of the Laplacian matrix

Signed Laplacian Matrix of a Signed Graph

- For our signed graph $G = (V, E, w)$ with n nodes, the Signed Laplacian is:

$$L(G) := D - A$$

- D is the degree matrix:

$$D_{ij} = \begin{cases} |\{a : (v_i, a) \in E\}|, & i = j \\ 0, & else \end{cases}$$

- A is the signed adjacency matrix:

$$A_{ij} = \begin{cases} 1, & (v_i, v_j) \in E \ \& \ w((v_i, v_j)) = 1 \\ -1, & (v_i, v_j) \in E \ \& \ w((v_i, v_j)) = -1 \\ 0, & else \end{cases}$$

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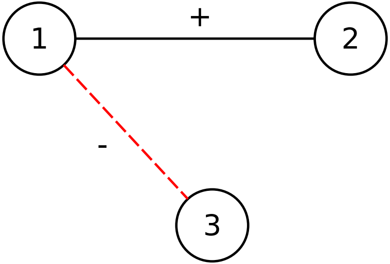
- Aside: The Laplacian matrix of general edge weighted undirected graphs is $L = D - A$ where D and A are the weighted degree and adjacency matrices respectively. This is a similar but fundamentally different definition than the Signed Laplacian

Signed Laplacian Matrix of a Signed Graph

Consider the following graph G:



$$D = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad L(G) = D - A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Properties of the Signed Laplacian

- $L = D - A$, therefore L is a real symmetric matrix
- By Spectral Theorem we therefore have an orthonormal eigenbasis $\mathbf{b}_1, \mathbf{b}_2, \dots \mathbf{b}_n \in \mathbb{R}^{|V|}$ with corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

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  - $\forall \mathbf{x} \in \mathbb{R}^{|V|} : \exists w_i \in \mathbb{R} : \mathbf{x} = \sum_{i=1}^{|V|} w_i \mathbf{b}_i$
- It can also be shown that the signed Laplacian is also positive semi-definite
  - $\forall \mathbf{x} : \mathbf{x}^T L \mathbf{x} \geq 0$

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From positive semi-definiteness, we know that $\lambda_1 \geq 0$ (Exercise: Prove this!). But why do we care about the eigenvalues of the signed Laplacian?

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Theorem

For a signed graph G, let λ_1 be the smallest eigenvalue of the corresponding signed Laplacian, $L(G)$. Then G is (completably) strongly balanced iff $\lambda_1 = 0$.

- Furthermore, it can be shown that signed graphs that are "close" to being balanced have "small" values of λ_1

Finding large balanced subgraphs

- We can show that for the Signed Laplacian $L(G)$ with smallest eigenvalue λ_1, then $\lambda_1 = 0$ iff G is strongly balanced
- There is a result indicating that graphs which are "close" to being balanced have "small' values of λ_1
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- Greedy approach: Repeatedly remove the nodes that cause the greatest decrease in λ_1 until the graph becomes strongly balanced
- This is the approach used by Ordozgoiti et al. (see `https://arxiv.org/abs/2002.00775`)

Finding large balanced subgraphs

- Let $\lambda_1(M)$ denote the smallest eigenvalue of the matrix M
- As calculating λ_1 is too expensive to be done $|V|$ times per removed node. Ordozgoiti et al. instead calculate $\lambda_1(L(G))$, and approximate λ_1 when choosing which node to remove from G
- Through a simple (but a bit long) derivation, the authors show that:

$$\lambda_1(L^{(i)}) \leq \frac{\lambda_1(L) + (\mathbf{b}_1)_i^2(d(i) - 2\lambda_1(L(G))) - \sum_{j\in\mathcal{N}(i)}(\mathbf{b}_1)_j^2}{1 - (\mathbf{b}_1)_i^2}$$

- In the above: $L^{(i)}$ is the signed Laplacian after the removal of the node v_i, $\mathbf{b}_1$ is the first eigenvector of $L(G)$, $\mathcal{N}(i)$ are the neighbours of the node v_i, and $d(i)$ is the degree of the node v_i.
- The derivation is straightforwards but a bit long, the details can be found in the paper

Finding large balanced subgraphs

- The author's algorithm uses this bound to greedily remove nodes until a balanced subgraph is found
- After a balanced subgraph is found, we check if the removed nodes can be re-introduced

Finding large balanced subgraphs

Algorithm 1 TIMBAL Algorithm

Input: signed graph G
$R \leftarrow \varnothing$
while G is not balanced **do**
  Compute $L(G)$, $\lambda_1(L(G))$, and corresponding $\mathbf{b}_1$
  $k \leftarrow \arg\min_i \frac{\lambda_1(L)+(\mathbf{b}_1)_i^2(d(i)-2\lambda_1(L(G)))-\sum_{j\in\mathcal{N}(i)}(\mathbf{b}_1)_j^2}{1-(\mathbf{b}_1)_i^2}$
  $G \leftarrow$ largest connected component in $G \setminus \{v_k\}$
  $R \leftarrow R \cup \{v_k\}$
end while
for $v \in R$ **do**
  if $G \cup \{v\}$ is balanced **then**
    $G \leftarrow G \cup \{v\}$
  end if
end for
return G

Finding large balanced subgraphs

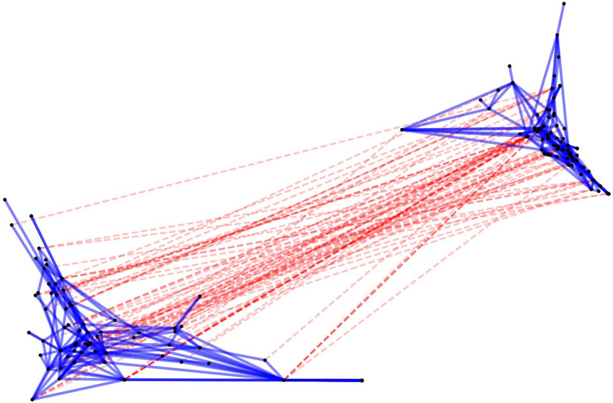
Table 2: Largest balanced subgraph found by each method for each dataset

	HighlandTribes		Cloister		Congress		Bitcoin		TwitterReferendum	
method	$\|V\|$	$\|E\|$	$\|V\|$	$\|E\|$	$\|V\|$	$\|E\|$	$\|V\|$	$\|E\|$	$\|V\|$	$\|E\|$
Timbal	13	35	10	33	208	452	4 208	10 158	8 944	166 243
Grasp	10	18	6	11	115	145	2 167	3 686	5 425	49 105
Ggmz	10	21	5	7	153	238	1 388	1 683	2 501	2 821
Eigen	12	37	8	27	11	16	7	17	132	6 140
	WikiElections		Slashdot		WikiConflict		WikiPolitics		Epinions	
Timbal	3 786	18 550	42 205	96 460	48 136	356 204	63 252	218 360	62 010	169 894
Grasp	1 752	4 416	23 289	40 511	18 576	82 726	31 561	81 557	28 189	63 250
Ggmz	713	771	16 389	17 867	6 137	9 145	23 342	37 098	21 009	25 013
Eigen	11	41	35	491	11	28	10	45	6	14

[Table from Ordozgoiti]

- Under various optimizations, the algorithm is able to process the Epinions dataset (containing 1 millions nodes and 12 million edges) in 1.5 hours

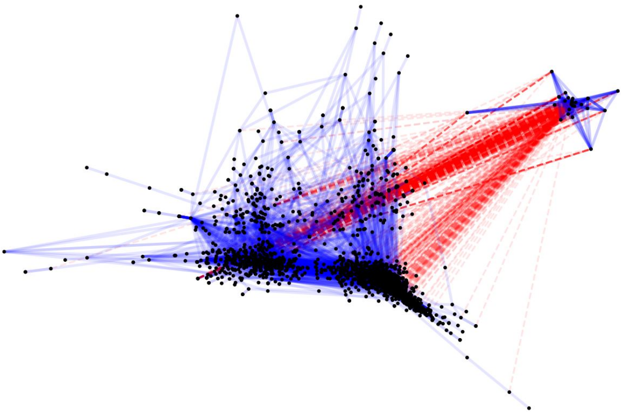
Finding large balanced subgraphs



[Figure from Ordozgoiti]

- Identified subgraph in the Congress dataset
- Edges represent (un)favourable mentions

Finding large balanced subgraphs



[Figure from Ordozgoiti]

- Identified subgraph in the Bitcoin OTC dataset
- Edges represent declared trust/distrust

Recap

- Structural Balance
  - Balanced triangles
  - Strongly balanced networks
  - Strong balance theorem
  - Weak structural balance
  - The signed Laplacian matrix