Social and Information Networks

Tutorial #4: Approximately Balanced Networks

University of Toronto CSC303

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Week 5: Feb 7-11 (2022)

Today's agenda

In lecture we've covered Chapter 5 of the textbook looking at Structural Balance.

Today:

- Questions from Lecture
- Recap of Structural Balance
- Approximately Balanced Networks (Ch 5.5b of E&K)
- Quercus Quiz

Questions?

Recap: Stable Triangles

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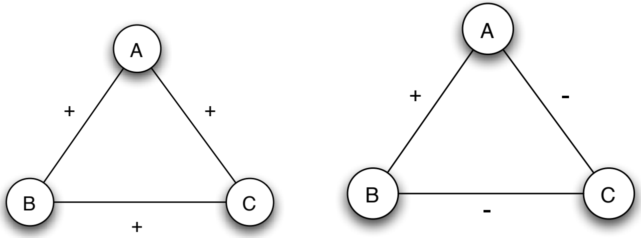
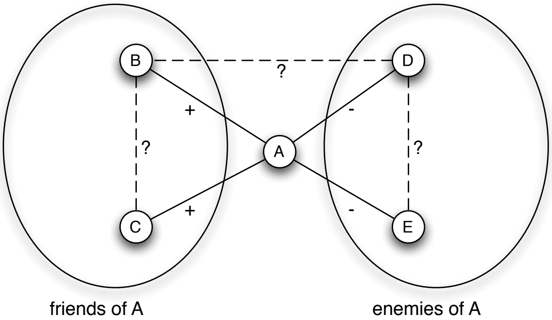


Recap: Harary Balance Theorem

If a labelled complete graph is balanced, then either everyone is friends, or the nodes can be partitioned into 2 groups that mutually loath each other and are internally purely friendly.



Recap: Testing if a graph can be completed into a balanced graph

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1. Find the positive communities (i.e. BFS on only the positive edges to find the positive connected components)
2. Confirm that these do not contain any negative edges
3. Collapse the positive communities into supernodes (and we collapse the negative edges between these communities accordingly)
4. Check that the graph of supernodes connected by negative edges is bipartite (via modified BFS)

Approximately Balanced Networks

Harary Balance Theorem only works when there are exactly no unbalanced triangles. Can we weaken this result to allow for some imbalanced triangles?

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Theorem

Let ϵ such that $0 \leq \epsilon \leq \frac{1}{8}$, and $\delta := \sqrt[3]{\epsilon}$. If at least $1 - \epsilon$ of the triangles in a complete labeled graph $G = (V, E)$ are balanced, then either:

1. *$\exists V' \subseteq V$ such that $\frac{|V'|}{|V|} \geq 1 - \delta$ and the proportion of hostile pairs in $|V'|$ is at most δ*
2. *We can partition V into X and Y such that:*
    - *The proportion of pairs in X that are hostile is at most δ*
    - *The proportion of pairs in Y that are hostile is at most δ*
    - *The proportion of edges between X & Y that are friendly is at most δ*

Proof

Can we modify the Proof of the Harary Balance Theorem? What problems could there be?

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The proof from lecture relies on dividing the graph based on the friends & enemies of a node A. Intuitively, we want to choose a node that is involved in a small number of violations. How do were formalize this intuition?

Proof

Let's look at the number of violations that a node is involved in. We know that the proportion of violating triangles out of all triangles is at most ϵ. Let $N := |V|$. How many triangles do we have?

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$$
{}_N C_3 = \binom{N}{3} = \frac{N(N-1)(N-2)}{6}
$$

Therefore we have at most $\epsilon \frac{N(N-1)(N-2)}{6}$ violating triangles.

Proof

Let $\mathrm{viol} : V \to \mathbb{N}$ count the number of violations that a node is involved in. Each triangle will be counted 3 times, therefore:

$$\sum_{v \in V} \mathrm{viol}(v) \leq 3 \times \epsilon \frac{N(N-1)(N-2)}{6}$$

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Note that we immediately know that $\exists A \in V$ such that:

$$\text{viol}(A) \leq 3 \times \epsilon \frac{N(N-1)(N-2)}{6} / |V| = \epsilon \frac{(N-1)(N-2)}{2} \leq \epsilon N^2/2$$

Let's use this node A to partition V into the sets X (A's friends) and Y (A's enemies).

Case 1: $|X| \geq (1-\delta)N$

Note $\delta = \sqrt[3]{\epsilon} < \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$, therefore $|X| > \frac{1}{2}N$.

Assuming N is even then $|X| \geq \frac{1}{2}N + 1$.

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Therefore, the number of edges between nodes in X is
$\binom{|X|}{2} \geq \binom{\frac{1}{2}N+1}{2} = (\frac{1}{2}N+1)(\frac{1}{2}N)/2 \geq (\frac{1}{2}N)^2/2 = N^2/8$

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Therefore, the proportion of edges between nodes in X causing a violation is at most:

$$\frac{\text{viol}(A)}{\binom{|X|}{2}} \leq \frac{\epsilon N^2/2}{N^2/8} = 4\epsilon = 4\delta^3 < \delta$$

Therefore we satisfy the theorem. Note that the final inequality holds as $\delta < \frac{1}{2}$.

Case 2: $|Y| \geq (1-\delta)N$

The same proof as Case 1 applies.

Case 3: $|X| < (1-\delta)N$ and $|Y| < (1-\delta)N$

As X and Y partition V, we know that $|X| + |Y| = |V| = N$, therefore $|X| = N - |Y| > \delta N$.

Assuming $\delta N \in \mathbb{N}$ then the number of edges within X is:

$$\binom{|X|}{2} \geq \binom{\delta N + 1}{2} > \delta^2 N^2/2$$

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Assuming $\delta N \in \mathbb{N}$ then the number of edges within X is:

$$\binom{|X|}{2} \geq \binom{\delta N + 1}{2} > \delta^2 N^2/2$$

As we showed before, any unfriendly edge within X will cause an unstable triangle with A, and $\mathrm{viol}(A) \leq \epsilon N^2/2$, therefore the proportion of unfriendly edges in X is at most:

$$\frac{\mathrm{viol}(A)}{\binom{|X|}{2}} \leq \frac{\epsilon N^2/2}{\delta^2 N^2/2} = \delta$$

The same argument holds for unfriendly edges in Y. All that remains is to show the same bound on the proportion of edges between X and Y that are friendly.

Case 3: $|X| < (1-\delta)N$ and $|Y| < (1-\delta)N$ cont'd

The number of edges between X and Y is $|X||Y| = |X|(N - |X|)$ for $\delta N < |X| < (1-\delta)N$.

As this is a concave quadratic function maximized at $|X| = 0.5N$ and $\delta < 0.5$, it's clear $|X|(N - |X|) > \delta(1-\delta)N^2 > \delta(1-0.5)N^2 = \delta N^2/2$.

Case 3: $|X| < (1-\delta)N$ and $|Y| < (1-\delta)N$ cont'd

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Therefore, using the same bound on violations as before we can see that the proportion of edges between X and Y that are friendly is at most:

$$\frac{\text{viol}(A)}{|X||Y|} \leq \frac{\epsilon N^2/2}{\delta N^2/2} = \delta^2 < \delta$$

Therefore X and Y satisfy the theorem.

Quercus Quiz