

X6.1 You are given natural variable n and the equation

$P \equiv \text{if } n \neq 1 \text{ then } n := \text{div } n \ 2. \ P \text{ else ok fi}$

where $\text{div } n \ 2$ is n divided by 2 and rounded down.

- (a) Find the first three specifications produced by recursive construction (no time variable).
- (b) What is the weakest specification P satisfying the equation? (No proof required.)

After trying the question, scroll down to the solution.

(a) Find the first three specifications produced by recursive construction (no time variable).

$$\begin{aligned}
 \S \quad P_0 &= \top \\
 P_1 &= \mathbf{if} \, n \neq 1 \, \mathbf{then} \, n := \text{div } n \, 2. \, \top \, \mathbf{else} \, ok \, \mathbf{fi} && \text{substitution law, expand } ok \\
 &= \mathbf{if} \, n \neq 1 \, \mathbf{then} \, \top \, \mathbf{else} \, n' = n \, \mathbf{fi} && \text{one case} \\
 &= n = 1 \Rightarrow n' = n && \text{context} \\
 &= n = 1 \Rightarrow n' = 1 && \text{context} \\
 P_2 &= \mathbf{if} \, n \neq 1 \, \mathbf{then} \, n := \text{div } n \, 2. \, n = 1 \Rightarrow n' = 1 \, \mathbf{else} \, ok \, \mathbf{fi} && \text{substitution law, expand } ok \\
 &= \mathbf{if} \, n \neq 1 \, \mathbf{then} \, \text{div } n \, 2 = 1 \Rightarrow n' = 1 \, \mathbf{else} \, n' = n \, \mathbf{fi} && \text{arithmetic, context} \\
 &= \mathbf{if} \, n \neq 1 \, \mathbf{then} \, n = 2 \vee n = 3 \Rightarrow n' = 1 \, \mathbf{else} \, n' = 1 \, \mathbf{fi} && \text{case analysis} \\
 &= (n \neq 1 \Rightarrow (n = 2 \vee n = 3 \Rightarrow n' = 1)) \wedge (n = 1 \Rightarrow n' = 1) && \text{portation and absorption} \\
 &= (n = 2 \vee n = 3 \Rightarrow n' = 1) \wedge (n = 1 \Rightarrow n' = 1) && \text{antidistributive} \\
 &= (n: 1, 2, 3 \Rightarrow n' = 1)
 \end{aligned}$$

(b) What is the weakest specification P satisfying the equation? (No proof required.)

$$\S \quad n \geq 1 \Rightarrow n' = 1$$