X5.5 A three-sided die has sides marked 0, 1, and 2 with probability $1/3$ for each side. Roll the die 3 times and add up the results. Calculate the distribution of the sum.

After trying the question, scroll down to the solution.

§ In this problem, it's quickest and easiest to just count the ways each result is produced. Letting the result be s' , we get

$$((s'=0)\times1 + (s'=1)\times3 + (s'=2)\times6 + (s'=3)\times7 + (s'=4)\times6 + (s'=5)\times3 + (s'=6)\times1)/27$$

But the point of this exercise is to write a program that expresses the distribution. The program is

$$s:=\ rand\ 3.\ \ s:= s + rand\ 3.\ \ s:= s + rand\ 3$$

We must replace the deceptive *rand* notation with a mathematical notation.

$(s'\colon 0,..3)/3.\ \ (s'\colon s+(0,..3))/3.\ \ (s'\colon s+(0,..3))/3$ first sequential composition

$= (\Sigma s''\cdot\ (s''\colon 0,..3)/3 \times (s'\colon s''+(0,..3))/3).\ \ (s'\colon s+(0,..3))/3$ sum

$= 1/3\times(s'\colon 0,..3)/3 + 1/3\times(s'\colon 1,..4)/3 + 1/3\times(s'\colon 2,..5)/3.\ \ (s'\colon s+(0,..3))/3$

$= (s'=0)\times1/9 + (s'=1)\times2/9 + (s'=2)\times3/9 + (s'=3)\times2/9 + (s'=4)\times1/9.\ \ (s'\colon s+(0,..3))/3$
sequential composition

$= \Sigma s''\cdot\ \ ((s''=0)\times1/9 + (s''=1)\times2/9 + (s''=2)\times3/9 + (s''=3)\times2/9 + (s''=4)\times1/9) \times (s'\colon s''+(0,..3))/3$ sum

$= 1/9\times(s'\colon (0,..3))/3 + 2/9\times(s'\colon 1+(0,..3))/3 + 3/9\times(s'\colon 2+(0,..3))/3 + 2/9\times(s'\colon 3+(0,..3))/3 + 1/9\times(s'\colon 4+(0,..3))/3$

$= (s'=0)\times1/27 + (s'=1)\times3/27 + (s'=2)\times6/27 + (s'=3)\times7/27 + (s'=4)\times6/27 + (s'=5)\times3/27 + (s'=6)\times1/27$

We were able to calculate the distribution because the die was rolled only 3 times. If it had been rolled a large number of times, or an indeterminate number, say n times, we would have written the program

$P \Longrightarrow\ \ s:= 0.\ \ i:= 0.\ \ Q$

$Q \Longrightarrow$ **if** $i=n$ **then** *ok* **else** $s:= s + rand\ 3.\ \ i:= i+1.\ \ Q$ **fi**

and we would have to define P and Q .