509 From the fixed-point equation

$$twos \;=\; c!\,2.\;\; t := t+1.\;\; twos$$

use recursive construction to find

(a) the weakest fixed-point.

(b) a strongest implementable fixed-point.

(c) the strongest fixed-point.

After trying the question, scroll down to the solution.

§ $twos \;=\; \mathcal{M}_{w}{=}2 \wedge \mathcal{T}_{w}{=}t \wedge w'{=}w{+}1 \wedge r'{=}r \wedge t'{=}t{+}1.\ twos$

(a) the weakest fixed-point.

§ If we start with $\top$, then

$twos_n \;=\; \forall i{:}\ 0,..n{\cdot}\ \mathcal{M}_{w+i}{=}2 \wedge \mathcal{T}_{w+i}{=}t{+}i$

$twos_\infty \;=\; \forall i{:}\ nat{\cdot}\ \mathcal{M}_{w+i}{=}2 \wedge \mathcal{T}_{w+i}{=}t{+}i$

(b) a strongest implementable fixed-point.

§ $twos \;=\; \mathcal{M}_{w}{=}2 \wedge \mathcal{T}_{w}{=}t \wedge w'{=}w{+}1 \wedge r'{=}r \wedge t'{=}t{+}1.\ twos$

$twos_0 \;=\; w'{\geq}w \wedge r'{\geq}r \wedge t'{\geq}t$

$twos_1 \;=\; \mathcal{M}_{w}{=}2 \wedge \mathcal{T}_{w}{=}t \wedge w'{\geq}w{+}1 \wedge r'{\geq}r \wedge t'{\geq}t{+}1$

$twos_2 \;=\; \mathcal{M}_{w}{=}2 \wedge \mathcal{T}_{w}{=}t \wedge \mathcal{M}_{w+1}{=}2 \wedge \mathcal{T}_{w+1}{=}t{+}1 \wedge w'{\geq}w{+}2 \wedge r'{\geq}r \wedge t'{\geq}t{+}2$

$twos_3 \;=\; \mathcal{M}_{w}{=}2 \wedge \mathcal{T}_{w}{=}t \wedge \mathcal{M}_{w+1}{=}2 \wedge \mathcal{T}_{w+1}{=}t{+}1 \wedge \mathcal{M}_{w+2}{=}2 \wedge \mathcal{T}_{w+2}{=}t{+}2$
$\wedge\ w'{\geq}w{+}3 \wedge r'{\geq}r \wedge t'{\geq}t{+}3$

$twos_n \;=\; (\forall i{:}\ 0,..n{\cdot}\ \mathcal{M}_{w+i}{=}2 \wedge \mathcal{T}_{w+i}{=}t{+}i) \wedge w'{\geq}w{+}n \wedge r'{\geq}r \wedge t'{\geq}t{+}n$

$twos_\infty \;=\; (\forall i{:}\ nat{\cdot}\ \mathcal{M}_{w+i}{=}2 \wedge \mathcal{T}_{w+i}{=}t{+}i) \wedge w'{=}\infty \wedge r'{=}r \wedge t'{=}\infty$

This is both implementable and deterministic, so it is a strongest implementable fixed-point.

(c) the strongest fixed-point.

§ If we start with $\bot$, then

$twos_n \;=\; twos_\infty \;=\; \bot$