432 From the axioms of program queue theory (Subsection 7.1.4), prove

(a) $front'=3 \Leftarrow mkempty.\ join\ 3$

(b) $front'=4 \Leftarrow mkempty.\ join\ 3.\ join\ 4.\ leave$

After trying the question, scroll down to the solution.

§ The axioms of program Queue Theory are

(0) $isemptyq' \Leftarrow mkemptyq$

(1) $isemptyq \Rightarrow front'=x \land \neg isemptyq' \Leftarrow join\ x$

(2) $\neg isemptyq \Rightarrow front'=front \land \neg isemptyq' \Leftarrow join\ x$

(3) $isemptyq \Rightarrow (join\ x.\ leave = mkemptyq)$

(4) $\neg isemptyq \Rightarrow (join\ x.\ leave = leave.\ join\ x)$

(a) $front'=3 \Leftarrow mkempty.\ join\ 3$

§ $mkempty.\ join\ 3$ — use (0) and (1) and monotonicity of .

$\Rightarrow\ isemptyq'.\ isemptyq \Rightarrow front'=3 \land \neg isemptyq'$ — use definition of .

$=\ \exists isemptyq'', front''\cdot\ isemptyq'' \land (isemptyq'' \Rightarrow front'=3 \land \neg isemptyq')$ — discharge

$=\ \exists isemptyq'', front''\cdot\ isemptyq'' \land front'=3 \land \neg isemptyq'$ — specialization and monotonicity of $\exists$

$\Rightarrow\ \exists isemptyq'', front''\cdot\ front'=3$ — unused quantifier

$=\ front'=3$

(b) $front'=4 \Leftarrow mkempty.\ join\ 3.\ join\ 4.\ leave$

§ Plan: Use (4) to commute (*join* 4. *leave*) . Then use (3) to change (*join* 3. *leave*) into *mkemptyq* . Then make (*mkemptyq*. *mkemptyq*) into *mkemptyq* . Then use (0) and (1) to make (*mkemptyq*. *join* 4) into $front'=4$.

(5) $\top$ — (1)

$=\ (isemptyq \Rightarrow front'=x \land \neg isemptyq' \Leftarrow join\ x)$ — portation

$=\ join\ x \land isemptyq \Rightarrow front'=x \land \neg isemptyq'$ — specialize

$\Rightarrow\ join\ x \land isemptyq \Rightarrow \neg isemptyq'$

(6) $\top$ — (2)

$=\ (\neg isemptyq \Rightarrow front'=front \land \neg isemptyq' \Leftarrow join\ x)$ — portation

$=\ join\ x \land \neg isemptyq \Rightarrow front'=front \land \neg isemptyq'$ — specialize

$\Rightarrow\ join\ x \land \neg isemptyq \Rightarrow \neg isemptyq'$

(7) $\top$ — (5) and (6)

$=\ (join\ x \land isemptyq \Rightarrow \neg isemptyq') \land (join\ x \land \neg isemptyq \Rightarrow \neg isemptyq')$ — antidistributive

$=\ join\ x \land isemptyq \lor join\ x \land \neg isemptyq \Rightarrow \neg isemptyq'$ — distributive

$=\ join\ x \land (isemptyq \lor \neg isemptyq) \Rightarrow \neg isemptyq'$ — excluded middle and identity

$=\ join\ x \Rightarrow \neg isemptyq'$ — inclusion

$=\ (join\ x = join\ x \land \neg isemptyq')$

(9) $a \Rightarrow (b=c)$ — antisymmetry

$=\ a \Rightarrow (b \Rightarrow c) \land (c \Rightarrow b)$ — distributive

$=\ (a \Rightarrow (b \Rightarrow c)) \land (a \Rightarrow (c \Rightarrow b))$ — specialize

$\Rightarrow\ a \Rightarrow (b \Rightarrow c)$ — portation

$=\ a \land b \Rightarrow c$ — symmetry

$=\ b \land a \Rightarrow c$ — portation

$=\ b \Rightarrow (a \Rightarrow c)$

(10) $\top$ — (4)

$=\ \neg isemptyq \Rightarrow (join\ x.\ leave = leave.\ join\ x)$ — (9)

$\Longrightarrow$ $(\mathit{join}\ x.\ \mathit{leave}) \Rightarrow (\neg \mathit{isemptyq} \Rightarrow (\mathit{leave}.\ \mathit{join}\ x))$

(11) $\top$ (3)
$\Longrightarrow$ $(\mathit{join}\ x.\ \mathit{leave}) \Rightarrow (\mathit{isemptyq} \Rightarrow \mathit{mkemptyq})$

Now the main proof.

	$\mathit{mkempty}.\ \mathit{join}\ 3.\ \mathit{join}\ 4.\ \mathit{leave}$	use (10) and monotonicity of .
$\Longrightarrow$	$\mathit{mkempty}.\ \mathit{join}\ 3.\ \neg \mathit{isemptyq} \Rightarrow (\mathit{leave}.\ \mathit{join}\ 4)$	use (8) and monotonicity of .
$\Longrightarrow$	$\mathit{mkempty}.\ \mathit{join}\ 3 \wedge \neg \mathit{isemptyq}'.\ \neg \mathit{isemptyq} \Rightarrow (\mathit{leave}.\ \mathit{join}\ 4)$	condition law
$\Longrightarrow$	$\mathit{mkempty}.\ \mathit{join}\ 3.\ \mathit{leave}.\ \mathit{join}\ 4$	use (11)
$\Longrightarrow$	$\mathit{mkempty}.\ \mathit{isemptyq} \Rightarrow \mathit{mkemptyq}.\ \mathit{join}\ 4$	use (0) twice
$\Longrightarrow$	$\mathit{isemptyq}'.\ \mathit{isemptyq} \Rightarrow \mathit{isemptyq}'.\ \mathit{join}\ 4$	use definition of . on first .
$=$	$(\exists \mathit{isemptyq}'', \mathit{front}''\cdot\ \mathit{isemptyq}'' \wedge (\mathit{isemptyq}'' \Rightarrow \mathit{isemptyq}')).\ \mathit{join}\ 4$	discharge
$=$	$(\exists \mathit{isemptyq}'', \mathit{front}''\cdot\ \mathit{isemptyq}'' \wedge \mathit{isemptyq}').\ \mathit{join}\ 4$	one-pt and unused
$=$	$\mathit{isemptyq}'.\ \mathit{join}\ 4$	use (1)
$=$	$\mathit{isemptyq}'.\ \mathit{isemptyq} \Rightarrow \mathit{front}'{=}4 \wedge \neg \mathit{isemptyq}'$	use definition of .
$=$	$\exists \mathit{isemptyq}'', \mathit{front}''\cdot\ \mathit{isemptyq}'' \wedge (\mathit{isemptyq}'' \Rightarrow \mathit{front}'{=}4 \wedge \neg \mathit{isemptyq}')$	discharge
$=$	$\exists \mathit{isemptyq}'', \mathit{front}''\cdot\ \mathit{isemptyq}'' \wedge \mathit{front}'{=}4 \wedge \neg \mathit{isemptyq}'$	one-pt and unused
$=$	$\mathit{front}'{=}4 \wedge \neg \mathit{isemptyq}'$	specialize
$=$	$\mathit{front}'{=}4$	

I hope there's a shorter proof.