416 Using the definition of Exercise 415, but ignoring time, prove

(a) $\textbf{do } P \textbf{ while } b \textbf{ od} \;=\; P.\; \textbf{while } b \textbf{ do } P \textbf{ od}$

(b) $\textbf{while } b \textbf{ do } P \textbf{ od} \;=\; \textbf{if } b \textbf{ then do } P \textbf{ while } b \textbf{ od else } ok \textbf{ fi}$

(c) $(\forall\sigma, \sigma'\cdot\ D = \textbf{do } P \textbf{ while } b \textbf{ od}) \wedge (\forall\sigma, \sigma'\cdot\ W = \textbf{while } b \textbf{ do } P \textbf{ od})$

$= \quad (\forall\sigma, \sigma'\cdot\ (D = P.\ W)) \wedge (\forall\sigma, \sigma'\cdot\ W = \textbf{if } b \textbf{ then } D \textbf{ else } ok \textbf{ fi})$

After trying the question, scroll down to the solution.

(a) $\quad \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ =\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}$

§ $\quad \forall\sigma, \sigma'\cdot (\mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \Leftarrow\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})$ reflexive and identity

$=\quad (\quad \forall\sigma, \sigma'\cdot (P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \Leftarrow\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}))$ use **while** construction

$\Leftarrow\quad (\quad \forall\sigma, \sigma'\cdot (P.\ \mathbf{if}\ b\ \mathbf{then}\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi}\ \Leftarrow\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \Leftarrow\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})\ )$

let $D\ =\ (P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})$

$=\quad (\quad \forall\sigma, \sigma'\cdot (P.\ \mathbf{if}\ b\ \mathbf{then}\ D\ \mathbf{else}\ ok\ \mathbf{fi}\ \Leftarrow\ D)$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \Leftarrow\ D)\ )$ **do** induction axiom

$=\quad \top$

That's half of what we want. For the other half,

$\quad \forall\sigma, \sigma'\cdot (P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od})$ Use **do** construction

$\Leftarrow\quad \forall\sigma, \sigma'\cdot (P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ P.\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

sequential composition is monotonic

$\Leftarrow\quad \forall\sigma, \sigma'\cdot (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

reflexive and identity and use **do** construction

$=\quad (\quad \forall\sigma, \sigma'\cdot (\quad \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi}$

$\quad\quad \Leftarrow\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})\ )$

$\Leftarrow\quad (\quad \forall\sigma, \sigma'\cdot (\quad \mathbf{if}\ b\ \mathbf{then}\ P.\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi}\ \mathbf{else}\ ok\ \mathbf{fi}$

$\quad\quad \Leftarrow\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})\ )$

let $W\ =\ (\mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

$=\quad (\quad \forall\sigma, \sigma'\cdot (\mathbf{if}\ b\ \mathbf{then}\ P.\ W\ \mathbf{else}\ ok\ \mathbf{fi}\ \Leftarrow\ W)$

$\quad \Rightarrow\ \forall\sigma, \sigma'\cdot (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \Leftarrow\ W)\ )$ **while** induction axiom

$=\quad \top$

which is the other half of what we want.
An almost identical proof can be made from fixed-point axioms.

(b) $\quad \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ =\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi}$

§ $\quad (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ =\ \mathbf{if}\ b\ \mathbf{then}\ \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$ use part (a)

$=\quad (\mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ =\ \mathbf{if}\ b\ \mathbf{then}\ P.\ \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od}\ \mathbf{else}\ ok\ \mathbf{fi})$

while fixed-point construction

$=\quad \top$

(c) $\quad (\forall\sigma, \sigma'\cdot D = \mathbf{do}\ P\ \mathbf{while}\ b\ \mathbf{od}) \wedge (\forall\sigma, \sigma'\cdot W = \mathbf{while}\ b\ \mathbf{do}\ P\ \mathbf{od})$

$=\quad (\forall\sigma, \sigma'\cdot (D = P.\ W)) \wedge (\forall\sigma, \sigma'\cdot W = \mathbf{if}\ b\ \mathbf{then}\ D\ \mathbf{else}\ ok\ \mathbf{fi})$