409 Let i be an extended integer variable in the refinement

$$P \Leftarrow \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i{:=}\ i-1.\ \ t{:=}\ t+1.\ \ P \textbf{ fi}$$

Investigate how recursive construction works when we start with

(a) $t' = \infty$

(b) $t{:=}\ \infty$

(c) ok

After trying the question, scroll down to the solution.

(a) $t' = \infty$

§ $P_0 = t'=\infty$

$P_1 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ t'=\infty \textbf{ fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } t'=\infty \textbf{ fi}$

$P_2 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } t'=\infty \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else if } i=1 \textbf{ then } i'=i-1 \wedge t'=t+1 \textbf{ else } t'=\infty \textbf{ fi fi}$

$= \textbf{if } 0 \le i < 2 \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

$P_n = \textbf{if } 0 \le i < n \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

$P_\infty = \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

Now we need to test whether P_∞ is a solution for P.

$\textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ P_\infty \textbf{ fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i:= i-1.\ t:= t+1.\ \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else if } 1 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi fi}$

$= \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } t'=\infty \textbf{ fi}$

$= P_\infty$

(b) $t:= \infty$

§ $P_0 = t:= \infty$

$P_1 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ t:= \infty \textbf{ fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i'=i-1 \wedge t'=\infty \textbf{ fi}$

$P_2 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i'=i-1 \wedge t'=\infty \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else if } i=1 \textbf{ then } i'=i-1 \wedge t'=t+1 \textbf{ else } i'=i-2 \wedge t'=\infty \textbf{ fi fi}$

$= \textbf{if } 0 \le i < 2 \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-2 \wedge t'=\infty \textbf{ fi}$

$P_n = \textbf{if } 0 \le i < n \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-n \wedge t'=\infty \textbf{ fi}$

$P_\infty = \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-\infty \wedge t'=\infty \textbf{ fi}$

We cannot simplify $i-\infty$ to $-\infty$ because i is an extended integer and $\infty-\infty$ cannot be simplified to $-\infty$.

Now we need to test whether P_∞ is a solution for P.

$\textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ P_\infty \textbf{ fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t$

$\textbf{else } i:= i-1.\ t:= t+1.\ \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-\infty \wedge t'=\infty \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else if } 1 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-\infty \wedge t'=\infty \textbf{ fi fi}$

$= \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-\infty \wedge t'=\infty \textbf{ fi}$

$= P_\infty$

If i were an integer variable (not extended), it would not be clear what P_∞ is.

(c) ok

§ $P_0 = ok$

$P_1 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ ok \textbf{ fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i'=i-1 \wedge t'=t+1 \textbf{ fi}$

$P_2 = \textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i'=i-1 \wedge t'=t+1 \textbf{ fi fi}$

$= \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else if } i=1 \textbf{ then } i'=i-1 \wedge t'=t+1 \textbf{ else } i'=i-2 \wedge t'=t+2 \textbf{ fi fi}$

$= \textbf{if } 0 \le i < 2 \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-2 \wedge t'=t+2 \textbf{ fi}$

$P_n = \textbf{if } 0 \le i < n \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-n \wedge t'=t+n \textbf{ fi}$

$P_\infty = \textbf{if } 0 \le i < \infty \textbf{ then } i'=0 \wedge t'=t+i \textbf{ else } i'=i-\infty \wedge t'=\infty \textbf{ fi}$

Now we need to test whether P_∞ is a solution for P.

$\textbf{if } i=0 \textbf{ then } ok \textbf{ else } i:= i-1.\ t:= t+1.\ P_\infty \textbf{ fi}$

$=$ **if** $i=0$ **then** $i'=i \wedge t'=t$
else $i:= i-1.\ t:= t+1.$ **if** $0 \le i < \infty$ **then** $i'=0 \wedge t'=t+i$ **else** $i'=i-\infty \wedge t'=\infty$ **fi fi**
$=$ **if** $i=0$ **then** $i'=i \wedge t'=t$ **else if** $1 \le i < \infty$ **then** $i'=0 \wedge t'=t+i$ **else** $i'=i-\infty \wedge t'=\infty$ **fi fi**
$=$ **if** $0 \le i < \infty$ **then** $i'=0 \wedge t'=t+i$ **else** $i'=i-\infty \wedge t'=\infty$ **fi**
$=$ P_∞

If i were an integer variable (not extended), it would not be clear what P_∞ is.