406 Let i be an integer variable, and let P be a specification such that

$$P \;=\; i{:=}\ i-1.\ \ \textbf{if}\ i{=}0\ \textbf{then}\ i{:=}\ 3\ \textbf{else}\ P.\ \ i{:=}\ 3\ \textbf{fi}$$

(a) Add recursive time.

(b) Is $i'{=}3 \wedge (0{<}i \Rightarrow t'{=}t{+}i{-}1)$ a fixed-point (solution for P)? Prove or disprove.

After trying the question, scroll down to the solution.

(a) Add recursive time.

§ $P = i:= i-1.\ \textbf{if } i=0 \textbf{ then } i:= 3 \textbf{ else } t:= t+1.\ P.\ i:= 3 \textbf{ fi}$

(b) Is $i'=3 \wedge (0<i \Rightarrow t'=t+i-1)$ a fixed-point (solution for P)? Prove or disprove.

§ Starting with the right side,

$i:= i-1.\ \textbf{if } i=0 \textbf{ then } i:= 3 \textbf{ else } t:= t+1.\ i'=3 \wedge (0<i \Rightarrow t'=t+i-1).\ i:= 3 \textbf{ fi}$ — factor

$= i:= i-1.\ \textbf{if } i=0 \textbf{ then } ok \textbf{ else } t:= t+1.\ i'=3 \wedge (0<i \Rightarrow t'=t+i-1) \textbf{ fi}.\ i:= 3$ — substitution

$= i:= i-1.\ \textbf{if } i=0 \textbf{ then } i'=i \wedge t'=t \textbf{ else } i'=3 \wedge (0<i \Rightarrow t'=t+i) \textbf{ fi}.\ i:= 3$ — substitution

$= \textbf{if } i=1 \textbf{ then } i'=i-1 \wedge t'=t \textbf{ else } i'=3 \wedge (1<i \Rightarrow t'=t+i-1) \textbf{ fi}.\ i:= 3$ — expand final asmt

$= \textbf{if } i=1 \textbf{ then } i'=i-1 \wedge t'=t \textbf{ else } i'=3 \wedge (1<i \Rightarrow t'=t+i-1) \textbf{ fi}.\ i'=3 \wedge t'=t$ — seq comp

$= \exists i'', t''\cdot\ \textbf{if } i=1 \textbf{ then } i''=i-1 \wedge t''=t \textbf{ else } i''=3 \wedge (1<i \Rightarrow t''=t+i-1) \textbf{ fi} \wedge i'=3 \wedge t'=t''$ — one-point for t''

$= \exists i''\cdot\ \textbf{if } i=1 \textbf{ then } i''=i-1 \wedge t'=t \textbf{ else } i''=3 \wedge (1<i \Rightarrow t'=t+i-1) \textbf{ fi} \wedge i'=3$ — factor $i'=3$ outside $\exists i''\cdot$ (distributive law)

$= i'=3 \wedge \exists i''\cdot\ \textbf{if } i=1 \textbf{ then } i''=i-1 \wedge t'=t \textbf{ else } i''=3 \wedge (1<i \Rightarrow t'=t+i-1) \textbf{ fi}$ — case analysis

$= i'=3 \wedge \exists i''\cdot\ (i=1 \wedge i''=i-1 \wedge t'=t) \vee (i\ne 1 \wedge i''=3 \wedge (1<i \Rightarrow t'=t+i-1))$ — splitting

$= i'=3 \wedge ((\exists i''\cdot\ i=1 \wedge i''=i-1 \wedge t'=t) \vee (\exists i''\cdot\ i\ne 1 \wedge i''=3 \wedge (1<i \Rightarrow t'=t+i-1)))$ — one-point twice

$= i'=3 \wedge ((\underline{i=1} \wedge t'=t) \vee (i\ne 1 \wedge (1<i \Rightarrow t'=t+i-1)))$ — context

$= i'=3 \wedge ((i=1 \wedge t'=t+i-1) \vee (i\ne 1 \wedge \underline{(1<i \Rightarrow t'=t+i-1)}))$ — inclusion

$= i'=3 \wedge ((i=1 \wedge t'=t+i-1) \vee \underline{(i\ne 1 \wedge (1\ge i \vee t'=t+i-1))})$ — distribute

$= i'=3 \wedge ((i=1 \wedge t'=t+i-1) \vee \underline{(i\ne 1 \wedge 1\ge i)} \vee (i\ne 1 \wedge t'=t+i-1))$ — simplify middle disjunct

$= i'=3 \wedge (\underline{(i=1 \wedge t'=t+i-1)} \vee i<1 \vee \underline{(i\ne 1 \wedge t'=t+i-1)})$ — first and last disjunct

$= i'=3 \wedge \underline{(i<1 \vee t'=t+i-1)}$ — inclusion

$= i'=3 \wedge (0<i \Rightarrow t'=t+i-1)$

So yes, it is a solution. I expect there's a shorter proof.