

- 40 (family theory) Design a theory of personal relationships. Invent person expressions such as *Jack*, *Jill*, *father of p*, *mother of p*. Invent binary expressions that use person expressions, such as *p is male*, *p is female*, *p is a parent of q*, *p is a son of q*, *p is a daughter of q*, *p is a child of q*, *p is married to q*,  $p=q$ . Invent axioms such as  $(p \text{ is male}) \neq (p \text{ is female})$ . Formulate and prove an interesting theorem.

After trying the question, scroll down to the solution.

§ I don't like the axiom  $(p \text{ is male}) \neq (p \text{ is female})$ . I'll add the binary expressions  $p \text{ is alive}$  and  $p \text{ is dead}$ . Here are some axioms.

- (a)  $p \text{ is a child of } q \equiv q \text{ is a parent of } p$
- (b)  $p \text{ is a parent of } q \equiv p = \text{mother of } q \vee p = \text{father of } q$
- (c)  $p \text{ is a child of } q \equiv p \text{ is a son of } q \vee p \text{ is a daughter of } q$
- (d)  $(p \text{ is alive}) \neq (p \text{ is dead})$

And it's easy to think of many more. Here is a well-known little story. A father and his son are in an accident; the father is killed, and the son is injured. The son is rushed to the hospital where the surgeon declares: "I can't operate on this boy because he's my son.". The point of this story is that you find it puzzling if you are afflicted with sexist assumptions. We will prove that the surgeon is the boy's mother. Introducing *the boy* and *the surgeon* as two new person expressions, the given information is

- (0) *the father of the boy is dead*
- (1) *the surgeon is alive*
- (2) *the boy is a son of the surgeon*

The proof starts with a little lemma.

$$\begin{aligned}
 \text{(e)} \quad & (a \equiv b \vee c) && \text{antisymmetry} \\
 = & (a \Rightarrow b \vee c) \wedge (b \vee c \Rightarrow a) && \text{specialization} \\
 \Rightarrow & b \vee c \Rightarrow a && \text{antidistributive} \\
 = & (b \Rightarrow a) \wedge (c \Rightarrow a) && \text{specialization} \\
 \Rightarrow & b \Rightarrow a
 \end{aligned}$$

Now the main proof.

$$\begin{aligned}
 & \top && \text{axiom (c)} \\
 = & (p \text{ is a child of } q \equiv p \text{ is a son of } q \vee p \text{ is a daughter of } q) \\
 & \text{this is of the form } a \equiv b \vee c, \text{ so use lemma (e)} \\
 \Rightarrow & p \text{ is a son of } q \Rightarrow p \text{ is a child of } q && \text{instantiate } p := \text{the boy. } q := \text{the surgeon} \\
 \Rightarrow & \text{the boy is a son of the surgeon} \Rightarrow \text{the boy is a child of the surgeon} \\
 & \text{use (2) and direct proof} \\
 \Rightarrow & \text{the boy is a child of the surgeon} && \text{use (a)} \\
 = & \text{the surgeon is a parent of the boy} && \text{use (b)} \\
 = & \text{the surgeon} = \text{the mother of the boy} \vee \text{the surgeon} = \text{the father of the boy} \\
 & \text{transparency} \\
 \Rightarrow & \text{the surgeon} = \text{the mother of the boy} \\
 & \vee \text{the surgeon is alive} = \text{the father of the boy is alive} && \text{use (1) and (0) and (d)} \\
 = & \text{the surgeon} = \text{the mother of the boy} \\
 \end{aligned}$$

which is the desired result.