373 Prove the law $nat = 0,..\infty$ from the other laws.

After trying the question, scroll down to a solution attempt.

§ I'll start by proving half of it.

$nat: 0,..\infty$

In this half, I am trying to prove nat isn't too big, so I'll need nat induction

$0, B+1: B \;\Rightarrow\; nat: B$

Here we go:

	$nat: 0,..\infty$	use induction with $0,..\infty$ for B
$\Leftarrow$	$0, (0,..\infty)+1:\; 0,..\infty$	addition distributes over bunch union
$=$	$0, 0+1,..\infty+1:\; 0,..\infty$	arithmetic; ∞ absorbs additions
$=$	$0, 1,..\infty:\; 0,..\infty$	
$=$	$0,..\infty:\; 0,..\infty$	reflexive
$=$	$\top$	

The other half is

$0,..\infty: nat$

In this half, I am trying to prove that all of $0,..\infty$ is in nat, so I should need nat construction.

$0, nat+1 : nat$

So maybe I should prove

$0,..\infty: 0, nat+1$

and then the result will follow by transitivity. Also, these laws look like they might be useful:

$x, y: xint \;\land\; x{\le}y \;\Rightarrow\; (i: x,..y \;=\; i: xint \;\land\; x{\le}i{<}y)$	interval law
$A: B \;=\; \forall x: A\cdot\, x: B$	inclusion law
$V: W \;=\; \forall v: V\cdot\, \exists w: W\cdot\; v{=}w$	bunch-element conversion law
$A: B \;\land\; B: C \;\Rightarrow\; A: C$	transitivity

But I am unable to find a proof.

Here's an attempt to prove both halves together.

	$\top$	identity
$=$	$\forall i: xint\cdot\, \top$	interval law
$=$	$\forall i: xint\cdot\, x, y: xint \;\land\; x{\le}y \;\Rightarrow\; (i: x,..y \;=\; i: xint \;\land\; x{\le}i{<}y)$	specialize
$\Rightarrow$	$\forall i: xint\cdot\, 0, \infty: xint \;\land\; 0{\le}\infty \;\Rightarrow\; (i: 0,..\infty \;=\; i: xint \;\land\; 0{\le}i{<}\infty)$	antecedent all $\top$
$=$	$\forall i: xint\cdot\, (i: 0,..\infty \;=\; i: xint \;\land\; 0{\le}i{<}\infty)$	context provided by quantification
	(quantifier applies to a function; function variable introduction is axiom in body)	
$=$	$\forall i: xint\cdot\, (i: 0,..\infty \;=\; \top \;\land\; 0{\le}i{<}\infty)$	identity
$=$	$\forall i: xint\cdot\, (i: 0,..\infty \;=\; 0{\le}i{<}\infty)$	I can't justify this step
$=$	$\forall i: xint\cdot\, (i: 0,..\infty \;=\; i: nat)$	I can't justify this step either
$=$	$\forall i: xint\cdot\, 0,..\infty \;=\; nat$	idempotent
$=$	$0,..\infty \;=\; nat$	

You might just say it's obvious that $nat = 0,..\infty$, so why do we have to prove it? We have an application for our math: formal methods of software development. For that application, we want some binary expressions to be theorems (for example, $nat = 0,..\infty$), and we want other binary expressions to be antitheorems (for example, $nat \ne 0,..\infty$), and there are other binary expressions that we don't care about (for example, $0/0 = 1$). If we say $nat = 0,..\infty$ is obvious, that just means it's obvious that we want it to be a theorem. If it cannot be proven, we need to add it to the theory. I have added it, but it may be provable from the other laws.