355 (biased Monty Hall) There are three doors numbered 0, 1, and 2. Monty rolls a 6-sided die, and if it shows 1, 2, or 3 Monty places the prize behind door 0; if it shows 4 or 5 Monty places the prize behind door 1, and if it shows 6 Monty places the prize behind door 2. The contestant chooses a door. Monty then opens one of the doors, but not the door with the prize behind it, and not the door the contestant has chosen. Monty asks the contestant whether they (the contestant) would like to change their choice of door, or stay with their original choice. What should the contestant do?

After trying the question, scroll down to the solution.

§ Let p be the door where the prize is. That's

$$((p'=0)/2 + (p'=1)/3 + (p'=2)/6)$$

Suppose the contestant chooses door 0 and Monty opens door 1, which tells the contestant that the prize is not behind door 1.

$$p' \neq 1$$

We know two distributions, so we multiply them together. But multiplying distributions does not necessarily result in a distribution, so we must normalize the result to obtain a distribution.

$$\begin{aligned}
& \quad (((p'=0)/2 + (p'=1)/3 + (p'=2)/6)) \times (p' \neq 1) \\
& / ((\Sigma p' \cdot ((p'=0)/2 + (p'=1)/3 + (p'=2)/6) \times (p' \neq 1)) \\
= & \quad ((p'=0)\times(p' \neq 1)/2 + (p'=1)\times(p' \neq 1)/3 + (p'=2)\times(p' \neq 1)/6) \\
& / (1/2 + 0 + 1/6) \\
= & ((p'=0)/2 + (p'=2)/6) / (2/3) \\
= & (p'=0) \times 3/4 \ + \ (p'=2) \times 1/4
\end{aligned}$$

So the prize is more probably behind door 0, and the contestant should stick with their choice of door 0.

Suppose the contestant chooses door 0 and Monty opens door 2, which tells the contestant that the prize is not behind door 2.

$$p' \neq 2$$

We know two distributions, so we multiply them together and normalize.

$$\begin{aligned}
& \quad (((p'=0)/2 + (p'=1)/3 + (p'=2)/6)) \times (p' \neq 2) \\
& / ((\Sigma p' \cdot ((p'=0)/2 + (p'=1)/3 + (p'=2)/6) \times (p' \neq 2)) \\
= & \quad ((p'=0)\times(p' \neq 2)/2 + (p'=1)\times(p' \neq 2)/3 + (p'=2)\times(p' \neq 2)/6) \\
& / (1/2 + 1/3 + 0) \\
= & ((p'=0)/2 + (p'=1)/3) / (5/6) \\
= & (p'=0) \times 3/5 \ + \ (p'=1) \times 2/5
\end{aligned}$$

So the prize is again more probably behind door 0, and the contestant should stick with their choice of door 0.

Suppose the contestant chooses door 1 and Monty opens door 0. The resulting distribution is

$$\begin{aligned}
& \quad (((p'=0)/2 + (p'=1)/3 + (p'=2)/6)) \times (p' \neq 0) \\
& / ((\Sigma p' \cdot ((p'=0)/2 + (p'=1)/3 + (p'=2)/6) \times (p' \neq 0)) \\
= & \quad ((p'=0)\times(p' \neq 0)/2 + (p'=1)\times(p' \neq 0)/3 + (p'=2)\times(p' \neq 0)/6) \\
& / (0 + 1/3 + 1/6) \\
= & ((p'=1)/3 + (p'=2)/6) / (1/2) \\
= & (p'=1) \times 2/3 \ + \ (p'=2) \times 1/3
\end{aligned}$$

So the prize is more probably behind door 1, and the contestant should stick.

Suppose the contestant chooses door 1 and Monty opens door 2. The resulting distribution is

$$\begin{aligned}
& \quad (((p'=0)/2 + (p'=1)/3 + (p'=2)/6)) \times (p' \neq 2) \\
& / ((\Sigma p' \cdot ((p'=0)/2 + (p'=1)/3 + (p'=2)/6) \times (p' \neq 2)) && \text{same as before} \\
= & (p'=0) \times 3/5 \ + \ (p'=1) \times 2/5
\end{aligned}$$

So the prize is more probably behind door 0, and the contestant should switch.

Suppose the contestant chooses door 2 and Monty opens door 0. The resulting distribution is

$$(p'=1) \times 2/3 \ + \ (p'=2) \times 1/3$$

So the prize is more probably behind door 1, and the contestant should switch.

Suppose the contestant chooses door 2 and Monty opens door 1. The resulting distribution is

$(p'=0) \times 3/4 + (p'=2) \times 1/4$

So the prize is more probably behind door 0, and the contestant should switch.

In addition to finding out whether to stick or switch, we also find out that the contestant maximizes their chance of winning ($(2/3 + 3/4)/2 = 17/24$) by choosing door 2 and switching.