302 For what exact precondition and exact postcondition does the following assignment move integer variable x farther from zero staying on the same side of zero?

(a) $x:= x+1$

(b) $x:= abs\ (x+1)$

(c) $x:= x^2$

After trying the question, scroll down to the solution.

§ What does "staying on the same side of zero" mean if the initial value of x is zero? Since that's not clear, let's say that x' can be on either side in that case. The specification is

$$(x<0 \Rightarrow x'<x) \land (x=0 \Rightarrow x' \neq 0) \land (x>0 \Rightarrow x'>x)$$
$$= \quad x'<x<0 \lor x' \neq x=0 \lor x'>x>0$$

(a) $x:= x+1$

§ (exact precondition for $x'<x<0 \lor x' \neq x=0 \lor x'>x>0$ to be refined by $x:=x+1$)
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ (x:= x+1)$
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ x'=x+1$ One-point
$= \quad x+1<x<0 \lor x+1 \neq x=0 \lor x+1>x>0$
$= \quad \bot \lor x=0 \lor x>0$
$= \quad x \geq 0$

We can be sure that $x:= x+1$ will move x farther from zero, staying on the same side, if $x \geq 0$.

(exact postcondition for $x'<x<0 \lor x' \neq x=0 \lor x'>x>0$ to be refined by $x:=x+1$)
$= \quad \forall x \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ (x:= x+1)$
$= \quad \forall x \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ x'=x+1$ One-point
$= \quad x'<x'-1<0 \lor x' \neq x'-1=0 \lor x'>x'-1>0$
$= \quad \bot \lor x'=1 \lor x'>1$
$= \quad x' \geq 1$

We can be sure that $x:= x+1$ moved x farther from zero, staying on the same side, if we see $x' \geq 1$.

(b) $x:= abs\ (x+1)$

§ (exact precondition for $x'<x<0 \lor x' \neq x=0 \lor x'>x>0$ to be refined by $x:= abs\ (x+1)$)
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ (x:= abs\ (x+1))$
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ x'=abs\ (x+1)$ One-point
$= \quad abs\ (x+1)<x<0 \lor abs\ (x+1) \neq x=0 \lor abs\ (x+1)>x>0$
$= \quad \bot \lor x=0 \lor x>0$
$= \quad x \geq 0$

We can be sure that $x:= abs(x+1)$ will move x farther from zero, staying on the same side, if $x \geq 0$.

(exact postcondition for $x'<x<0 \lor x' \neq x=0 \lor x'>x>0$ to be refined by $x:= abs\ (x+1)$)
$= \quad \forall x \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ (x:= abs\ (x+1))$
$= \quad \forall x \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ x'=abs\ (x+1)$
$= \quad \forall x \cdot\ abs\ (x+1)<x<0 \lor abs\ (x+1) \neq x=0 \lor abs\ (x+1)>x>0 \ \Leftarrow\ x'=abs\ (x+1)$
$= \quad \forall x \cdot\ x \geq 0 \ \Leftarrow\ x'=abs\ (x+1)$
$= \quad x'<0$

We can be sure that $x:= abs\ (x+1)$ moved x farther from zero, staying on the same side, if we see $x'<0$. But of course we will see $x' \geq 0$, so we can never be sure that x moved farther from zero, staying on the same side.

(c) $x:= x^2$

§ (exact precondition for $x'<x<0 \lor x' \neq x=0 \lor x'>x>0$ to be refined by $x:= x^2$)
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ (x:= x^2)$
$= \quad \forall x' \cdot\ x'<x<0 \lor x' \neq x=0 \lor x'>x>0 \ \Leftarrow\ x'= x^2$ One-point
$= \quad x^2<x<0 \lor x^2 \neq x=0 \lor x^2>x>0$
$= \quad x \geq 2$

We can be sure that $x := x^2$ will move x farther from zero, staying on the same side, if $x \ge 2$.

$$
\begin{array}{ll}
 & (\text{exact postcondition for } x'<x<0 \lor x'{=}x{=}0 \lor x'>x>0 \text{ to be refined by } x:=x^2) \\
= & \forall x\cdot\ x'<x<0 \lor x'{=}x{=}0 \lor x'>x>0 \ \Leftarrow\ (x:=x^2) \\
= & \forall x\cdot\ x^2<x<0 \lor x^2{=}x{=}0 \lor x^2>x>0 \ \Leftarrow\ x'=x^2 \\
= & \forall x\cdot\ \bot \lor \bot \lor x>1 \ \Leftarrow\ x'=x^2 \\
= & \forall x\cdot\ \bot \lor \bot \lor x>1 \ \Leftarrow\ x'=x^2 \qquad \text{contrapositive} \\
= & \forall x\cdot\ x\le 1 \ \Rightarrow\ x'{\ne}x^2 \qquad \text{idempotence} \\
= & (\forall x\cdot\ x\le 1 \ \Rightarrow\ x'{\ne}x^2) \land (\forall x\cdot\ x\le 1 \ \Rightarrow\ x'{\ne}x^2) \\
 & \qquad \text{use a Change of Variable Law to change one } x \text{ to } -x \\
= & (\forall x\cdot\ x\le 1 \ \Rightarrow\ x'{\ne}x^2) \land (\forall x\cdot\ -x\le 1 \ \Rightarrow\ x'{\ne}(-x)^2) \\
= & (\forall x\cdot\ x\le 1 \ \Rightarrow\ x'{\ne}x^2) \land (\forall x\cdot\ x\ge -1 \ \Rightarrow\ x'{\ne}x^2) \qquad \text{combine} \\
= & \forall x\cdot\ x'{\ne}x^2
\end{array}
$$

We can be sure that $x := x^2$ moved x farther from zero, staying on the same side, if x' is not a square. But of course it will be a square, so we can never be sure that x moved farther from zero, staying on the same side.