277 Given three sorted lists having at least one item common to all three, write a program to find the smallest item occurring in all three lists.

After trying the question, scroll down to the solution.

§ Let the three lists be A , B , and C and let a , b , and c be natural variables to act as indexes of the three lists. The specification is R , defined as

$$\begin{aligned} R \;=\;\; & A(0,..a')\text{ ‘ }B(0,..b')\text{ ‘ }C(0,..c') = null \\ & \land\; A\,a' = B\,b' = C\,c' \\ & \land\; t' = t+a'+b'+c' \end{aligned}$$

which says there isn't a common item before indexes a' , b' , and c' , and there is one at those indexes, and the sum of those indexes is the execution time. We also need the specification Q , defined as

$$\begin{aligned} Q \;=\;\; & A(a,..a')\text{ ‘ }B(b,..b')\text{ ‘ }C(c,..c') = null \\ & \land\; A\,a' = B\,b' = C\,c' \\ & \land\; t' = t+a'-a+b'-b+c'-c \end{aligned}$$

which says the same thing, but starting at indexes a , b , and c . The refinements are

$R \;\Leftarrow\; a:= 0.\; b:= 0.\; c:= 0.\; Q$

$Q \;\Leftarrow\;$ **if** $A\,a < B\,b$ **then** $a:= a+1.\; t:= t+1.\; Q$
else if $B\,b < C\,c$ **then** $b:= b+1.\; t:= t+1.\; Q$
else if $C\,c < A\,a$ **then** $c:= c+1.\; t:= t+1.\; Q$
else ok **fi fi fi**

The proof of the first refinement is immediate after using the Substitution Law 3 times. The proof of the last refinement breaks into 12 pieces (3 conjuncts × 4 cases). Let's start with the first conjunct of Q and the first case of the **if** .

$A\,a < B\,b \land (a:= a+1.\; t:= t+1.\; A(a,..a')\text{ ‘ }B(b,..b')\text{ ‘ }C(c,..c') = null)$
substitution law twice
$=\;\; A\,a < B\,b \;\land\; A(a+1,..a')\text{ ‘ }B(b,..b')\text{ ‘ }C(c,..c') = null$
Because B is sorted, $B\,b$ is the minimum item of $B(b,..b')$.
And since $A\,a < B\,b$, therefore $A\,a$ is unequal to any item in $B(b,..b')$.
$\Rightarrow\;\; A(a,..a')\text{ ‘ }B(b,..b')\text{ ‘ }C(c,..c') = null$

Now we prove the middle conjunct of Q with the same first case of the **if** .

$A\,a < B\,b \land (a:= a+1.\; t:= t+1.\; A\,a' = B\,b' = C\,c')$ substitution law twice
$=\;\; A\,a < B\,b \;\land\; A\,a' = B\,b' = C\,c'$ specialize
$\Rightarrow\;\; A\,a' = B\,b' = C\,c'$

Now we prove the last conjunct of Q with the same first case of the **if** .

$A\,a < B\,b \land (a:= a+1.\; t:= t+1.\; t' = t+a'-a+b'-b+c'-c)$ substitution law twice
$=\;\; A\,a < B\,b \;\land\; t' = t+1+a'-a-1+b'-b+c'-c$ simplify and specialize
$\Rightarrow\;\; t' = t+a'-a+b'-b+c'-c$

The proofs of the next two cases are exactly the same. That leaves the last case.

$A\,a \ge B\,b \land B\,b \ge C\,c \land C\,c \ge A\,a \land ok$ antisymmetry, and expand ok
$A\,a = B\,b = C\,c \;\land\; a'=a \;\land\; b'=b \;\land\; c'=c \;\land\; t'=t$
$\Rightarrow\;\; Q$