222 (natural square root) The natural square root of a natural number n is the natural number s satisfying

$$s^2 \le n < (s+1)^2$$

(a) Write a program to find the natural square root of a given natural number n in $log\ n$ time.

(b) Write a program to find the natural square root of a given natural number n in $log\ n$ time using only addition, subtraction, doubling, halving, and comparisons (no multiplication or division).

After trying the question, scroll down to the solution.

(a) Write a program to find the natural square root of a given natural number n in $log\ n$ time.

§ The solution is a binary search for s in the interval $0,..n+1$.

$$s'^{\,2} \le n < (s'+1)^2 \ \Longleftarrow\ s:= 0.\ r:= n+1.\ s^2 \le n < r^2 \Rightarrow s'^{\,2} \le n < (s'+1)^2$$

$$s^2 \le n < r^2 \Rightarrow s'^{\,2} \le n < (s'+1)^2 \ \Longleftarrow$$

if $s+1 = r$ **then** ok
else $m:= div\ (s+r)\ 2.$
    if $m^2 \le n$ **then** $s:= m$ **else** $r:= m$ **fi**.
    $s^2 \le n < r^2 \Rightarrow s'^{\,2} \le n < (s'+1)^2$ **fi**

Execution time is identical to binary search.

(b) Write a program to find the natural square root of a given natural number n in $log\ n$ time using only addition, subtraction, doubling, halving, and comparisons (no multiplication or division).

§ The problem with the program in part (a) is the occurrence of m^2. We'll have to work in the squared space to avoid the need for squaring. Our three variables will be:

s is the left end of the remaining search space times the size of the remaining search space. The left end is initially 0, so s is initially 0. The size of the search space is finally 1, so s' is the answer.

b is the square of the size of the remaining search space. To cut the search space in half, we must divide b by 4. To repeatedly divide b by 4, it must be a power of 4.

n is the number whose square root is sought minus the square of the left end of the search space. Initially the left end is 0, so n starts as the number whose square root is sought. n measures, in squared space, the distance from s to the answer.

Define

$P\ =\ n'=n<b':b\times 4^{nat} \land s'=s \land$ **if** $b\le 4\times n$ **then** $b'\le 4\times n$ **else** $b'=b$ **fi**
$\quad\land\ t' = t + (log(b'/b))/2$

$Q\ =\ n < b+2\times s\ \land\ (n\le 2\times s \Rightarrow s=0)\ \land\ (\exists m, l\cdot\ b=4^m \land s=l\times 2^m)$
$\quad\Rightarrow\ b'=1\ \land\ n' = n-s'^2 \le 2\times s'\ \land\ t' = t + (log\ b)/2$

Then

$$s'^{\,2} \le n < (s'+1)^2\ \land\ t' \le t + log\ (n+1) + 2\ \Longleftarrow\ s:= 0.\ b:= 1.\ P.\ Q$$

$P \Longleftarrow$ **if** $b>n$ **then** ok **else** $b:= 2\times b.\ b:= 2\times b.\ t:= t+1.\ P$ **fi**

$Q \Longleftarrow$ **if** $b=1$ **then** ok
else $b:= b/2.\ b:= b/2.\ c:= s+b.\ s:= s/2.$
    if $c > n$ **then** ok **else** $(n:= n-c.\ s:= s+b)$ **fi**.
    $t:= t+1.\ Q$ **fi**