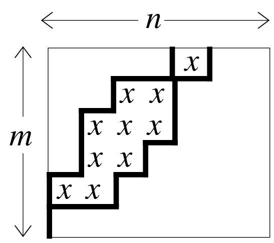
193 (sorted two-dimensional count) Write a program to count the number of occurrences of a given item in a given 2-dimensional array in which each row is sorted and each column is sorted. The execution time must be linear in the sum of the dimensions.

After trying the question, scroll down to the solution.

§ Let A be the array, let its size be $m\times n$, and let x be the item (these are constants). The solution will use variables r, a, b, c: *nat* . Variable r will indicate a row; on that row there may be some occurrences of item x; a will indicate the start of these occurrences, and b will indicate the end of these occurrences. Variable c will accumulate the count of the occurrences of x. As r decreases from m to 0, variables a and b will trace the envelope of the occurrences of x, as in the following picture.



The problem is P, defined as

$$P \;=\; c' = \Sigma i\colon 0,..m\cdot \Sigma j\colon 0,..n\cdot \textbf{if } A\,i\,j = x \textbf{ then } 1 \textbf{ else } 0 \textbf{ fi}$$

Define

$$\begin{aligned} Q \;=\;& \quad c = (\Sigma i\colon r,..m\cdot \Sigma j\colon 0,..n\cdot \textbf{if } A\,i\,j = x \textbf{ then } 1 \textbf{ else } 0 \textbf{ fi}) \\ &\wedge\ (\forall i\colon 0,..r\cdot \forall j\colon 0,..a\cdot A\,i\,j < x) \ \wedge\ (\forall i\colon r,..m\cdot \forall j\colon b,..n\cdot x < A\,i\,j) \\ &\Rightarrow\ P \\ L \;=\;& (\forall j\colon 0,..a\cdot A\,r\,j < x) \\ &\Rightarrow\ (\forall j\colon 0,..a'\cdot A\,r\,j < x)\ \wedge\ (A\,r\,a' \ge x\ \vee\ a'=n)\ \wedge\ r'=r\ \wedge\ b'=b\ \wedge\ c'=c \\ R \;=\;& (\forall j\colon 0,..b\cdot A\,r\,j \le x) \\ &\Rightarrow\ (\forall j\colon 0,..b'\cdot A\,r\,j \le x)\ \wedge\ (A\,r\,b' > x\ \vee\ b'=n)\ \wedge\ r'=r\ \wedge\ a'=a\ \wedge\ c'=c \end{aligned}$$

Or, alternatively, define

$$\begin{aligned} Q \;=\;& \quad (a=0\ \vee\ A\,r\,(a-1) < x)\ \wedge\ (b=0\ \vee\ A\,r\,(b-1) \le x) \\ &\Rightarrow\ c' = c + (\Sigma i\colon 0,..r\cdot \Sigma j\colon 0,..n\cdot \textbf{if } A\,i\,j = x \textbf{ then } 1 \textbf{ else } 0 \textbf{ fi}) \\ L \;=\;& (a'=0\ \vee\ A\,r\,(a'-1) < x)\ \wedge\ (A\,r\,a' \ge x\ \vee\ a'=n)\ \wedge\ r'=r\ \wedge\ b'=b\ \wedge\ c'=c \\ R \;=\;& (b'=0\ \vee\ A\,r\,(b'-1) \le x)\ \wedge\ (A\,r\,b' > x\ \vee\ b'=n)\ \wedge\ r'=r\ \wedge\ a'=a\ \wedge\ c'=c \end{aligned}$$

Now the refinements:

$$\begin{aligned} P \;&\Leftarrow\; r:= m.\ a:= 0.\ b:= 0.\ c:= 0.\ Q \\ Q \;&\Leftarrow\; \textbf{if } r=0 \textbf{ then } ok \textbf{ else } r:= r-1.\ L.\ R.\ c:= c+b-a.\ Q \textbf{ fi} \\ L \;&\Leftarrow\; \textbf{if } a=n \textbf{ then } ok \textbf{ else if } A\,r\,a \ge x \textbf{ then } ok \textbf{ else } a:= a+1.\ L \textbf{ fi fi} \\ R \;&\Leftarrow\; \textbf{if } b=n \textbf{ then } ok \textbf{ else if } A\,r\,b > x \textbf{ then } ok \textbf{ else } b:= b+1.\ R \textbf{ fi fi} \end{aligned}$$

For the time, put $t:= t+1$ before each of the three recursive calls, replace specification P with $t' \le t + m + 2\times n$, and replace all the other specifications Q, L, and R with $a\le n \wedge b\le n \Rightarrow t' \le t + r + m + 2\times n - a - b$.