172√ (maximum item) Write a program to find the maximum item in a list.

See book Subsection 8.0.1. See also the solution below, after trying the question.

§ Let the list be L (a constant), and I assume its items are numbers. Let m be a number variable; its final value will be the answer. Let i be a natural variable to index L. Let t be time measured recursively. The problem is R, where

$$R \;=\; m'=\Uparrow L \;\land\; t' = t+\#L$$

Define $Q \;=\; m' = m \uparrow \Uparrow L[i;..\#L] \;\land\; t' = t+\#L-i$

Then

$$R \;\Leftarrow\; m:= -\infty.\; i:= 0.\; Q$$

Proof:

$m:= -\infty.\; i:= 0.\; Q$ — substitution law, twice

$=\; m' = -\infty \uparrow \Uparrow L[0;..\#L] \;\land\; t' = t+\#L-0$ — simplify

$=\; Q$

Now to refine Q.

$$Q \;\Leftarrow\; \textbf{if } i=\#L \textbf{ then } ok \textbf{ else } m:= m \uparrow L\,i.\; i:= i+1.\; t:= t+1.\; Q \textbf{ fi}$$

Proof, by cases. First case:

$i=\#L \land ok$ — expand ok, and then use context to complicate m and t

$=\; i=\#L \land m' = m \uparrow \Uparrow L[i;..\#L] \;\land\; t' = t+\#L-i$ — specialize

$\Rightarrow\; Q$

Last case:

$i\ne\#L \land (m:= m \uparrow L\,i.\; i:= i+1.\; t:= t+1.\; Q)$ — substitution, 3 times

$=\; i\ne\#L \;\land\; m' = m \uparrow L\,i \uparrow \Uparrow L[i+1;..\#L] \;\land\; t' = t+1+\#L-(i+1)$ — simplify time

$=\; i\ne\#L \;\land\; m' = m \uparrow L\,i \uparrow \Uparrow L[i+1;..\#L] \;\land\; t' = t+\#L-i$ — move $L\,i$ inside $\Uparrow$

$=\; i\ne\#L \;\land\; m' = m \uparrow \Uparrow L[i;..\#L] \;\land\; t' = t+\#L-i$ — specialize

$\Rightarrow\; Q$

For a **for**-loop solution, define

$$F\,i \;=\; m' = m \uparrow \Uparrow L[i;..\#L] \;\land\; t' = t+\#L-i$$

Now we solve the problem as follows:

$$R \;\Leftarrow\; m:= -\infty.\; F\,0$$

Proof:

$m:= -\infty.\; F\,0$ — expand F, then substitution law

$=\; m' = -\infty \uparrow \Uparrow L[0;..\#L] \;\land\; t' = t+\#L-0$ — simplify

$=\; R$

The remaining problem $F\,0$ is the right form to solve with a **for**-loop.

$$F\,0 \;\Leftarrow\; \textbf{for } i:= 0;..\#L \textbf{ do } m:= m \uparrow L\,i.\; t:= t+1 \textbf{ od}$$

We must prove the two refinements that this abbreviates. First

$0\le i<\#L \land (m:= m \uparrow L\,i.\; t:= t+1.\; F(i+1))$ — expand $F(j+1)$

$=\; 0\le i<\#L \land (m:= m \uparrow L\,i.\; t:= t+1.\; m' = m \uparrow \Uparrow L[i+1;..\#L] \;\land\; t' = t+\#L-(i+1))$ — substitution law twice

$=\; 0\le i<\#L \;\land\; m' = m \uparrow L\,i \uparrow \Uparrow L[i+1;..\#L] \;\land\; t' = t+1+\#L-(i+1)$ — simplify

$=\; 0\le i<\#L \;\land\; m' = m \uparrow \Uparrow L[i;..\#L] \;\land\; t' = t+\#L-i$ — specialize

$\Rightarrow\; F\,i$

Last

$F(\#L)$

$=\; m' = m \uparrow \Uparrow L[\#L;..\#L] \;\land\; t' = t+\#L-\#L$

$=\; m' = m \uparrow -\infty \;\land\; t'=t$

$=\; ok$

Alternatively, we could have used the invariant form of **for**-loop law, but without the timing. Define

$$A\,i \;=\; m = \Uparrow L[0;..i]$$

Then

$$R \;\Leftarrow\; m:= -\infty.\; A\,0 \Rightarrow A'(\#L)$$
$$A\,0 \Rightarrow A'(\#L) \;\Leftarrow\; \textbf{for } i:= 0;..\#L \textbf{ do } A\,i \Rightarrow A'(i+1) \textbf{ od}$$
$$A\,i \Rightarrow A'(i+1) \;\Leftarrow\; m:= m \uparrow L\,i$$

The first and last of these must be proven (the middle one is a gift), and the proofs are a lot like the proofs we have just done.