109 Let f and g be functions from *nat* to *nat* .

(a) For what f do we have the theorem $f\,g = g$?

(b) For what f do we have the theorem $g\,f = g$?

After trying the question, scroll down to the solution.

(a) For what f do we have the theorem $f\,g = g$?

§ Equality $f\,g = g$ means, first, that the domains are equal.

$$\square(f\,g) = \square g$$
$$= \quad (\S x{:}\ \square g\cdot\ g\ x{:}\ \square f) = \square g$$
$$= \quad (\S x{:}\ nat\cdot\ g\ x{:}\ nat) = nat$$
$$= \quad (\S x{:}\ nat\cdot\ \top) = nat$$
$$= \quad nat = nat$$
$$= \quad \top$$

so that's no constraint. Equality also means that the results are equal.

$$\forall x{:}\ nat\cdot\ (f\,g)\ x = g\ x$$
$$= \quad \forall x{:}\ nat\cdot f\,(g\ x) = g\ x$$

So f must be the identity function on the range of g .

$$= \quad \forall x{:}\ g\ nat\cdot f\,x = x$$

(b) For what f do we have the theorem $g\,f = g$?

§ The domains of $g\,f$ and g must be equal, and they are both nat . The results must also be equal.

$$\forall x{:}\ nat\cdot\ (g\,f)\ x = g\ x$$
$$= \quad \forall x{:}\ nat\cdot\ g\ (f\,x) = g\ x$$

For any x such that $f\,x \neq x$, g must give the same result for both $f\,x$ and x . If f is the identity function, then $g\,f = g$. If g is a constant function, then $g\,f = g$.