104 (unicorns) The following statements are made.

All unicorns are white.
All unicorns are black.
No unicorn is both white and black.

Are these statements consistent? What, if anything, can we conclude about unicorns?

After trying the question, scroll down to the solution.

§ Let *unicorn* be all unicorns. Let *white* and *black* be predicates on unicorns. Then

All unicorns are white:

(a) $\forall u\colon unicorn\cdot\ white\ u$

All unicorns are black:

(b) $\forall u\colon unicorn\cdot\ black\ u$

No unicorn is both white and black:

(c) $\neg\exists u\colon unicorn\cdot\ white\ u \land black\ u$

Suppose we take (a), (b), and (c) as axioms.

$\top$ (a), (b), and (c) are axioms

$= (\forall u\colon unicorn\cdot\ white\ u) \land (\forall u\colon unicorn\cdot\ black\ u) \land (\neg\exists u\colon unicorn\cdot\ white\ u \land black\ u)$

Using a duality law (deMorgan) on (c), we can change it to a universal quantification:

$= (\forall u\colon unicorn\cdot\ white\ u) \land (\forall u\colon unicorn\cdot\ black\ u) \land (\forall u\colon unicorn\cdot\ \neg(white\ u \land black\ u))$

Now we can use a splitting law to combine the three main conjuncts

$= \forall u\colon unicorn\cdot\ (white\ u \land black\ u) \land \neg(white\ u \land black\ u)$ Law of Noncontradiction

$= \forall u\colon unicorn\cdot\ \bot$ one-case

$=$ **if** $unicorn{=}null$ **then** $\forall u\colon unicorn\cdot\ \bot$ **else** $\forall u\colon unicorn\cdot\ \bot$ **fi**

In **then**-part, use **if**-part as context, and quantifier law $\forall v\colon null\cdot\ b$.

In **else**-part, use negation of **if**-part as context, and idempotent law.

$=$ **if** $unicorn{=}null$ **then** $\top$ **else** $\bot$ **fi** there ought to be a law

$= unicorn{=}null$

If we are given (a), (b), and (c) as axioms, we must conclude that there are no unicorns.