Mitres and bevels

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Preliminaries

Suppose you have to join two boards to form an outside corner of $2 \times C$ (so C is half the corner angle), both sloped at an angle of S from the vertical. I've tried to sketch this situation below. You need to find the mitre angle, M (the difference between a perpendicular cut on the face of the board and the cut you'll make), and the bevel, B (the difference between a perpendicular cut on the edge of the board and the cut you'll make). If you know C and S, here's a recipe to find M and B that you can work out with a pencil and a calculator with trigonometric functions (or tables of trig ratios).

The key trig ratios are based on a right-angle triangle with height H, width W, hypotenuse (the length of the side opposite the right angle) Y, forming an angle from the horizontal of $\angle A$:

- The tangent of A (abbreviated as $\tan A$) is H/W. This is the same as rise-over-run or the slope.
- The sine of A (abbreviated as $\sin A$) is H/Y.
- The cosine of A (abbreviated as $\cos A$) is W/Y.
- All three quantities are tied together by Pythagoras' theorem which says that $H^2 + W^2 = Y^2$. Also, a bit of arithmetic shows that $\sin A / \cos A = \tan A$ (write down the formulas and multiply them through).

Figuring out the angles

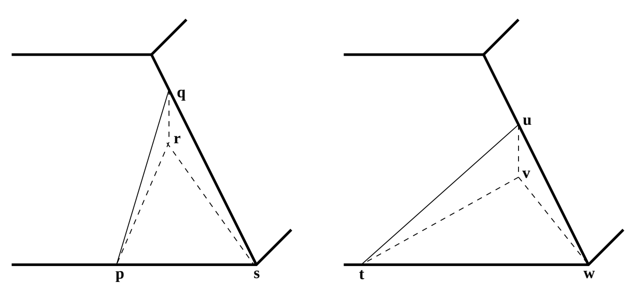

Figure 1: The sketch on the left is for calculating the mitre angle, M, and the one on the right is for calculating the bevel angle, B.

In the drawing on the left, put p one unit away from the corner s, and draw (or imagine drawing...) a line pq that is perpendicular to ps.. The angle $\angle pqs$ is the same as M that you're looking for.

Draw a line qr perpendicular to the horizontal working surface (this line is almost certainly imaginary). By the way you've constructed it, pr will perpendicular to both ps and qr. Now some calculation.

Since pr is perpendicular to ps, $\triangle rps$ is a right-angle triangle with angle $\angle psr$ the same as C, and (since ps is one unit long) $pr = \tan C$. To find the length of pq you can solve the triangle $\triangle pqr$ which has a right angle $\angle prq$, hypotenuse pq, and angle $\angle pqr = S$, so $\tan C/pq = \sin S$, or $pq = \tan C/\sin S$. Now $\triangle qps$ is a right triangle also, with ps one unit long, so $1/pq = \tan M$, or $pq = 1/\tan M$, so

$$a\frac{1}{\tan M} \quad = \quad \frac{\tan C}{\sin S} \Longrightarrow \qquad \tan M = \frac{sinS}{\tan C}. \tag{1}$$

In the drawing on the right, put t one unit away from the corner w, and draw tu perpendicular to uw. Draw a line uv perpendicular to the horizontal working surface, and by the way you've constructed it tv will be perpendicular to both tw and uv. Stare at $\triangle tuw$ until you convince yourself that it is similar (has the same three angles) as $\triangle pqs$ in the left-hand drawing — it probably doesn't look like in my drawing, but both triangles have a right angle, and share one angle, so their third angle must be the same. This means that $\angle utw = M$, and you've already calculated $\tan M$. You want to find B, which is the same as $\angle vtu$.

Since $\triangle tvw$ is a right-angle triangle, and you've made tw one unit long, $tv = \sin C$. Also, $\triangle tuw$ is a right-angle triangle and tw is one unit long, $tu = \cos M$. Since $\angle vtu = B$, you can now work out that

$$\cos B = \frac{tv}{tu} = \frac{\sin C}{\cos M}. \tag{2}$$

Since you already have a formula for $\tan M$, you have enough information to to get the angles you need (pocket calculator functions (or trig tables) arctan or $\tan^{-1}$ of $\tan M$ gives you the angle M, and arccos or $\cos^{-1}$ of $\cos B$ gives you the angle B), however you can boil things down a bit further with a bit of intense arithmetic.

To get an expression for $\cos M$ directly from $\tan M$, without converting to the angle M in between. To do this, use the Pythagorean formula on a right-angle triangle with height $\tan M$ and width 1, hypotenuse Y. Since this triangle makes horizontal angle M, you get $\tan^2 M + 1^2 = Y^2$, or $Y = \sqrt{\tan^2 M + 1}$, so $\cos M = 1/\sqrt{\tan^2 M + 1}$. Substitute in what you already know about $\tan M$, and

$$\frac{1}{\cos M} = \sqrt{\tan^2 M + 1} = \sqrt{\frac{\sin^2 S}{\tan^2 C} + 1} = \sqrt{\frac{\sin^2 S + \tan^2 C}{\tan^2 C}} = \frac{\sqrt{\sin^2 S + \tan^2 C}}{\tan C}.$$

Now substitute this expression for $1/\cos M$ into Equation ?? to get:

$$\begin{aligned} \cos B &= \sin C \times \frac{1}{\cos M} = \sin C \times \frac{\sqrt{\sin^2 S + \tan^2 C}}{\tan C} = \frac{\sin C}{\tan C}\sqrt{\sin^2 S + \tan^2 C} \\ &= \cos C\sqrt{\sin^2 S + \tan^2 C}. \end{aligned} \tag{3}$$

Formulas 1 and 3 give you all the information you need to find angles M and B.