Help for A1, week 3

June 4, 2004

Questions 1 and 2:

- You may discover a pattern, for example, that the number of segments in a length-n one-dimensional lattice is some sort of sum of the natural numbers up to n. Although you will get marks for a convincing argument that your pattern works, to get full marks you need to prove it. Similarly, you may notice that a length-$(n+1)$ lattice has a certain number more segments than a length-n lattice. You need to prove this observation, even if it seems blindingly obvious.
- Try counting the number of segments (squares, cubes) of a given size, see how that is connected to the size of the lattice, n.
- You may assume without proof that if k is a natural number, there are $k+1$ natural number there are $k+1$ natural numbers in the interval $[0, k]$.

Question 3:

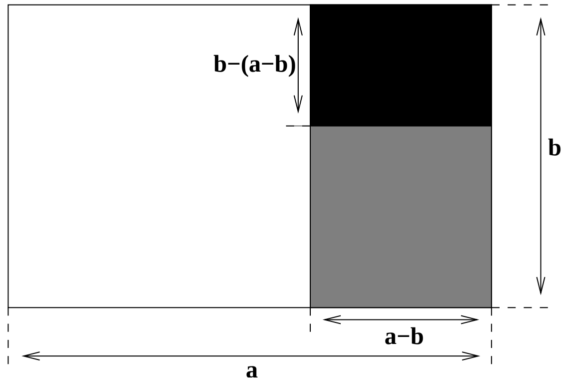
- You may use without proof the fact the a single point divides a line segment into two parts.
- You may use without proof the fact that the number of parts the plane is divided into by n lines is maximized when no two lines are parallel, and no three lines intersect in the same point.
- Consider the last (call it the nth) line that you draw in a configuration that maximizes the number of parts the plane is divided into. As it intersects each of the previous $n-1$ lines, it creates a new region of space and the intersection creates a point on that line. Look at the previous part of this question again.

Question 4:

You may use, without proof, the fact that the number of regions that n planes divide space into is maximized when no two planes are parallel, no three planes intersect in the same line, every set of three planes intersect in a single point, and no four planes intersect in the same point. If the number of three-dimensional regions is maximized, then each plane is divided up into the maximum number of two-dimensional regions by its intersections (lines) with the other planes. Look at the last question again.

Question 5:

- The following diagram makes it seem reasonable that if $a/b = b/(a-b)$, then $b/(a-b) = (a-b)/[b-(a-b)]$. What the algebra says is that if the rectangle with sides a and b is the same shape as the rectangle with sides b and $a-b$, then the rectangle with sides $a-b$ and $b-(a-b)$ also has the same shape. If a and b are natural numbers, then you can show you have a decreasing sequence of natural numbers b, $a-b$, $b-(a-b)$, $\ldots$, — is this possible? You will need to make the geometric intuition precise, perhaps using algebra.



- Although the question states the problem in terms of integers a and b, you can show that if there are integers that solve this problem, then there are natural numbers that solve it (equivalently, if there are no natural numbers that solve it, there are no integers that solve it). Furthermore, you can show that if there are natural numbers that solve the problem, they must be non-zero.

  If it makes the problem easier, assume that a and b must be positive natural numbers. If you state this assumption clearly, the marker will not deduct many marks.

Question 6:

- An n-digit binary number consisting entirely of ones can be expressed as the sum $\sum_{i=0}^{n-1} 2^i$ (you may use this without proving it). One convention for indicating that a number is in binary in binary is $(11111)_2$ (this is 31 in base 10).
- In the Week 3 Lectures notes there is (nearly) a proof that $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ (you can use this without proving it). Read the proof from Week 2's lecture that $12^n - 1$ is divisible by 11. Try to write down an inductive proof that $12^n - 1$ is divisible by 11 when n is even (then a proof for when n is odd).
- An n-digit ternary number consisting entirely of ones can be expressed as the sum $\sum_{i=0}^{n-1} 3^i$. Base ten 4 is written in ternary as $(11)_3$, which is the same as $3^1 + 3^0$.
- An n-digit ternary number consisting entirely of twos can be expressed as the sum $\sum_{i=0}^{n-1} 2 \times 3^i$ $= 2\sum_{i=0}^{n-1} 3^i$.

Question 7:

- A short induction proof shows that 10^n mod 9 equals 1 for all $n \in \mathbb{N}$.
- A short proof by induction shows that 10^n mod 11 equals 1 if n is even, or 10 if n is odd.