CSC236, Summer 2004, Assignment 3

Due: Thursday, July 15th, 10 pm

Instructions

Please work on all questions. Turn in the outline and structure of a proof, even if you cannot provide every step of the proof, and we will try to assign part marks. However, if there is any question you cannot see how to even begin, leave it blank you you will receive 20% of the marks for that question.

New policy: When you turn in your neat, finely-crafted, final version, indicate the question you feel you did the best work on by writing that question number on the front page of your assignment, and circling it. If we assign extra weight to any question, we will assign extra weight to the question you have circled.

Be sure to give full credit to any sources you consult (other than course notes, TAs, and the instructor) in preparing this problem set. If you try to pass off somebody else's work as your own for credit, you are committing an academic offense, and that can entail serious consequences. Any ideas that you do not attribute to someone else are assumed to be the ideas of the author(s) listed below, and will be evaluated for grading.

Write your name(s) and student number(s) (maximum of two names and two student numbers) in the space below, and staple this page to the front of your assignment.

Name ______________________________

Student # ______________________________

Name ______________________________

Student # ______________________________

1. Use structural induction to prove the following claims about propositional formulas.

   (a) Let $PV = \{P_1, \dots, P_k\}$ be a set of propositional variables, and F_{PV} be the set of all propositional formulas over PV, using Definition 5.1 on page 114 of the Course Notes. For any $e \in F_{PV}$, define $\mathbf{sf}(e)$ as the number of sub-formulas contained in e as substrings (including e itself). Define $\mathbf{pr}(e)$ as the number of parentheses in e, and define $\mathbf{ng}(e)$ as the number of unary connectives $\neg$ contained in e.

   CLAIM: For any $e \in F_{PV}$, $\mathbf{sf}(e) = \mathbf{pr}(e) + \mathbf{ng}(e) + 1$.

   (b) Define F^+_{PV} as the subset of formulas from part (a) that do NOT contain the unary connective $\neg$, and define $\mathbf{sf}(e)$ as in part (a).

   CLAIM: For any $e \in F^+_{PV}$, $\mathbf{sf}(e)$ is odd.

   (c) Define F_{PV} as in part (a), and for any $e \in F_{PV}$ define $\mathbf{par}(e)$ as $\mathbf{sf}(e) \bmod 2$.

   CLAIM: For any $e \in F_{PV}$, $\mathbf{par}(e) = (\mathbf{ng}(e) + 1) \bmod 2$.

2. In the propositional formulas below I use the rules of precedence from sections 5.2.4 and 5.6 of the Course Notes to reduce the number of parentheses. In your solution you are welcome to re-introduce parentheses if it makes things clearer.

   (a) Use the logical equivalences in section 5.6 (no truth tables) to prove that

   $$P \to Q \to R \to S \quad \text{LEQV} \quad \neg(P \wedge Q \wedge R \wedge \neg S).$$

   (b) Write a CNF formula equivalent to both formulas in part (a).

   (c) Use the logical equivalences in section 5.6 (no truth tables) to prove that

   $$P \leftrightarrow Q \leftrightarrow R \quad \text{LEQV} \quad P \wedge Q \wedge R \vee P \wedge \neg Q \wedge \neg R \vee \neg P \wedge \neg Q \wedge R \vee \neg P \wedge Q \wedge \neg R.$$

   (d) Use the logical equivalences from section 5.6, and part (c) of this assignment, to prove that $\leftrightarrow$ is associative, in other words prove that

   $$(P \leftrightarrow (Q \leftrightarrow R)) \quad \text{LEQV} \quad ((P \leftrightarrow Q) \leftrightarrow R)$$

3. Consider the truth table for $(P \oplus Q)$, on page 141 of the Course notes.

   (a) Write the truth table for $(P \oplus (Q \oplus R))$.

   (b) Write a CNF formula equivalent to $(P \oplus (Q \oplus R))$.

   (c) Write a DNF formula equivalent to $(P \oplus (Q \oplus R))$.

   (d) Is $\oplus$ associative? In other words, is the formula

   $$((P \oplus (Q \oplus R)) \leftrightarrow ((P \oplus Q) \oplus R))$$

   a tautology? Prove your claim.

4. Let $\mathcal{LA}^*$ be a first-order language with predicates L (of arity 2), $\approx$ (of arity 2, the equality predicate), and W (of arity 1), no constant symbols, and an infinite set of variables that include x, y, z, w. Consider the formula

   $$F: \qquad \exists z W(z) \to \exists y \forall x (W(y) \wedge (W(x) \to (L(y,x) \vee \approx (y,x))))$$

   (a) Does every interpretation of $\mathcal{LA}^*$ that has domain $D = \mathbb{N}$, and $L(z,w)$ interpreted as $z < w$ satisfy F? Explain your answer. (Notice that you must consider all possible interpretations of predicate W).

(b) Does every interpretation of $\mathcal{LA}^*$ that has domain $D = \mathbb{R}$ (the real numbers), and $L(z, w)$ interpreted as $z < w$ satisfy F? Explain your answer.

(c) Modify the formula by switching quantifiers:

$$F' : \qquad \exists z W(z) \rightarrow \forall x \exists y (W(y) \wedge (W(x) \rightarrow (L(y, x) \vee \approx (y, x)))).$$

Does every interpretation of $\mathcal{LA}^*$ that has domain $\mathbb{R}$ and $L(z, w)$ interpreted as $z < w$ satisfy F'? Explain your answer.

(d) What CSC236 concept does the original formula F express?

5. Let $\mathcal{LF}^+$ be the language of family relations with an infinite set of variables including x, y, and z, predicate symbols P (of arity 3), S (of arity 2), and $\approx$ (of arity 2, the equality predicate). Consider an interpretation where D is the set of all human beings, $P(x, y, z)$ means x and y are the parents of z, $S(x, y)$ means x is a sibling of y. State in careful English what it means for x to be a cousin of y. Then write a formula in $\mathcal{LF}^+$ that means that x is a cousin of y.