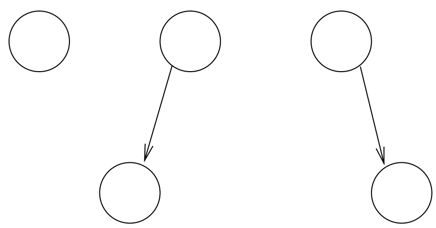
CSC236, Summer 2004, Assignment 2 Sample Solutions

1. Two unlabelled binary trees with zero nodes are considered the same. Two unlabelled binary trees with one node each are considered the same. Two unlabelled binary trees with more than one node each are considered the same if the left subtrees of their root nodes are the same, and the right subtrees of their root nodes are the same.

   The drawing below suggests there is one unlabelled binary tree with zero nodes, one unlabelled binary tree with one node and two unlabelled binary trees with two nodes. Derive a formula (it doesn't need to be in closed form) for the number of unlabelled binary trees there are with n nodes, for all $n \in \mathbb{N}$. Prove that your formula is correct.

   Claim: Define U_n as the set of unlabelled binary trees with n nodes, and define $B(n) = |U_n|$. Then

   $$B(n) = \begin{cases} 1, & n = 0 \\ \sum_{k=0}^{n-1} B(k)B(n-1-k) \end{cases}.$$

   Proof (complete induction on n): For $n = 0$, the definition of unlabelled binary trees says there is just one unlabelled binary tree with 0 nodes, so $B(0) = 1$. This verifies the base case. Induction step: Suppose $n > 0$, and assume that the claim holds for $0 \leq k < n$. The set U_n of unlabelled binary trees with n nodes can be partitioned into sets $U_{n,k}$ (where $0 \leq k \leq n-1$), where $U_{n,k}$ is defined as the set of unlabelled binary tree with n nodes and left sub-trees with k nodes. An unlabelled binary tree with a left subtree with k nodes must have $n-1-k$ nodes in its right subtree, so each element of $U_{n,k}$ matches exactly one element of U_k (its left sub-tree) with an element of U_{n-1-k} (its right subtree). Thus there is a 1–1 correspondence between the elements of $U_{n,k}$ and the elements of the Cartesian product $U_k \times U_{n-1-k}$, so

   $$\begin{aligned} B(n) = |U_n| &= \sum_{k=0}^{n-1} |U_{n,k}| = \sum_{k=0}^{n-1} |U_k \times U_{n-1-k}| = \sum_{k=0}^{n-1} |U_k|\,|U_{n-1-k}| \\ \text{[by IH]} &= \sum_{k=0}^{n-1} B(k)B(n-1-k). \end{aligned}$$

   Thus, whenever the claim holds for natural numbers $0 \leq k < n$ it also holds for n, as wanted. Conclude that the claim holds for all $n \in \mathbb{N}$. QED.

2. The FIBONACCI FUNCTION, $F(n)$ is defined in Example 3.2 (page 80) of the Course Notes. Verify each of the following claims for $n \in \{0, 1, 2\}$, and then prove that they are true for all $n \in \mathbb{N}$.

   (a) $\sum_{i=0}^{n} F(i) = F(n+2) - 1$.

   For $n = 0$, the claim says that $F(0) = F(2) - 1$, or $0 = 1 - 1$, which is true. For $n = 1$, the claim says that $F(0) + F(1) = F(3) - 1$, or $0 + 1 = 2 - 1$, which is true. For $n = 2$, the claim says that $F(0) + F(1) + F(2) = F(4) - 1$, or $0 + 1 + 1 = 3 - 1$, which is true. This verifies the claim for the first three natural numbers.

   PROOF (SIMPLE INDUCTION ON n): The base claim for $n = 0$ was verified in the paragraph above.

   INDUCTION STEP: I want to show that whenever the claim holds for a natural number n, then it also holds for $n + 1$, so I assume that the claim holds for an arbitrary natural number n. Now I can break up the sum $\sum_{i=0}^{n+1} F(i)$ and use the inductive hypothesis (IH):

$$\begin{aligned}
\sum_{i=0}^{n+1} F(i) &= \left[\sum_{i=0}^{n} F(i)\right] + F(n+1) \\
\text{[by IH]} &= F(n+2) - 1 + F(n+1) \\
\text{[definition of F(n+3), since } n+3 > 1\text{]} &= F(n+3) - 1 = F([n+1]+2) - 1.
\end{aligned}$$

   Thus, whenever the claim holds for an arbitrary natural number n, it also holds for $n + 1$, as wanted.

   I conclude that the claim holds for all $n \in \mathbb{N}$. QED.

   (b) $\sum_{i=0}^{n} F(2i) = F(2n+1) - 1$.

   For $n = 0$ the claim says that $F(0) = F(1) - 1$, or $0 = 1 - 1$, which is true. For $n = 1$, the claim says that $F(0) + F(2) = F(3) - 1$ or $0 + 1 = 2 - 1$, which is true. For $n = 2$, the claim says that $F(0) + F(2) + F(4) = F(5) - 1$, or $0 + 1 + 3 = 5 - 1$, which is true. This verifies the claim for the first three natural numbers.

   PROOF (INDUCTION ON n): The base claim for $n = 0$ was verified in the previous paragraph.

   INDUCTION STEP: I want to show that whenever the claim holds for an arbitrary natural number n, then it also holds for $n + 1$. I assume the claim holds for n, and then I can break up the sum $\sum_{i=0}^{n+1} F(2i)$ and use the inductive hypothesis (IH):

$$\begin{aligned}
\sum_{i=0}^{n+1} &= \left[\sum_{i=0}^{n} F(2i)\right] + F(2[n+1]) \\
\text{[by IH]} &= F(2n+1) - 1 + F(2[n+1]) = F(2n+1) + F(2n+2) - 1 \\
\text{[definition of } F(2n+3)\text{]} &= F(2n+3) - 1 = F(2[n+1]+1) - 1.
\end{aligned}$$

   Thus, if the claim holds for n it also holds for $n + 1$, as wanted.

   I conclude that the claim holds for all $n \in \mathbb{N}$. QED.

   (c) $\sum_{i=0}^{n} F(2i+1) = F(2n+2)$.

   When $n = 0$, the claim says that $F(1) = F(2)$, or $1 = 1$, which is true. When $n = 1$, the claim says that $F(1) + F(3) = F(4)$, or $1 + 2 = 3$, which is true. When $n = 2$, the claim says that $F(1) + F(3) + F(5) = F(6)$, or $1 + 2 + 5 = 8$, which is true. This verifies the claim for the first three natural numbers.

   PROOF (SIMPLE INDUCTION ON n): The base claim for $n = 0$ was verified in the previous paragraph.

   INDUCTION STEP: In order to show that whenever the claim holds for an arbitrary n it also holds for $n + 1$, I assume the claim holds for n. Now I can break up the sum $\sum_{i=0}^{n+1} F(2i+1)$

into two parts, and apply the inductive hypothesis (IH):

$$\begin{aligned}
\sum_{i=0}^{n+1} F(2i+1) &= \left[\sum_{i=0}^{n} F(2i+1)\right] + F(2[n+1]+1) \\
\text{[by IH]} &= F(2n+2) + F(2[n+1]+1) = F(2n+2) + F(2n+3) \\
\text{[definition of } F(2n+4)\text{]} &= F(2n+4) = F(2[n+1]+2).
\end{aligned}$$

Thus if the claim holds for n, it also holds for $n+1$, as wanted.
I conclude that the claim holds for all $n \in \mathbb{N}$. QED.

3. (a) On pages 88 and 89 a function $T(n)$ is defined to express the maximum number of steps required for a generic divide-and-conquer algorithm. Consider the case where n is a natural power of b, that is $n = b^k$ for some $k \in \mathbb{N}$, so

$$T(n) = \begin{cases} c, & n = 1 \\ aT\left(\frac{n}{b}\right) + dn^l, & n > 1. \end{cases}.$$

Prove that for any $0 \leq i \leq k$, you have

$$T(n) = a^i T\left(\frac{n}{b^i}\right) + dn^l \sum_{j=0}^{i-1} \left(\frac{a}{b^l}\right)^j.$$

This proof makes the informal reasoning near the bottom of page 89 precise. You may assume that c is a positive real number, that d, and l are non-negative real numbers, and that a and b are positive natural numbers.

SOLUTION: The notation seems easier to handle if I express n as a natural power of k, so suppose $n = b^k$. Then I can re-write $T(n)$

$$T(n) = T(b^k) = \begin{cases} c, & k = 0 \\ aT(b^{k-1}) + db^{kl}, & k > 0. \end{cases}$$

With this notation the claim becomes

CLAIM: $P(i)$ "If k is a natural number no less than i, then $T(b^k) = a^i T(b^{k-i}) + db^{kl} \sum_{j=0}^{i-1} (a/b^l)^j$" is true for all $i \in \mathbb{N}$.

PROOF (INDUCTION ON i): For $i = 0$ the claim says that $T(b^k) = a^0 T(b^k) = T(b^k)$, since the sum $\sum_{j=0}^{-1} (a/b^l)^j$ is empty. Thus the claim holds for the base case.

INDUCTION STEP: Assume the claim holds for some arbitrary $i \in \mathbb{N}$. If $i \geq k$, then $i + 1 > k$, so $P(i+1)$ holds, since the antecedent is false. The remaining case to consider is $0 \leq i < k$, which means $0 \leq i \leq k-1$, so I can unwind $T(b^k)$ and use the induction hypothesis for i and $T(b^{k-1})$:

$$\begin{array}{lrcl}
\text{definition of } T(b^k) & T(b^k) & = & aT(b^{k-1}) + db^{kl} \\
& \text{[by IH]} & = & a\left[a^i T\left(b^{k-1-i}\right) + db^{(k-1)l}\sum_{j=0}^{i-1}\left(\frac{a}{b^l}\right)^j\right] + db^{kl} \\
& & = & a^{i+1}T\left(b^{k-(i+1)}\right) + \frac{a}{b^l}db^{kl}\sum_{j=0}^{i-1}\left(\frac{a}{b^l}\right)^j + db^{kl}\left(a/b^l\right)^0 \\
& & = & a^{i+1}T\left(b^{k-(i+1)}\right) + db^{kl}\sum_{j=0}^{i-1}\left(\frac{a}{b^l}\right)^{j+1} + db^{kl}(a/b^l)^0 \\
& & = & a^{i+1}T\left(b^{k-(i+1)}\right) + db^{kl}\sum_{j=0}^{i}\left(\frac{a}{b^l}\right)^j \\
& & = & a^{i+1}T\left(b^{k-(i+1)}\right) + db^{kl}\sum_{j=0}^{(i+1)-1}\left(\frac{a}{b^l}\right)^j
\end{array}$$

Thus $P(i+1)$ holds whenever $P(i)$ holds, as wanted.
I conclude that $P(i)$ holds for every $i \in \mathbb{N}$. QED.

(b) Use simple induction to prove the principle of function definition by recursion (page 80, equation 3.5). To prove that function f exists, you must prove that $f(n)$ is defined for every $n \in \mathbb{N}$. To prove that $f(n)$ is unique, you must prove that any two functions that satisfy the definition also agree on their value for every $n \in \mathbb{N}$.

CLAIM: Suppose $b \in \mathbb{Z}$, and $G : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ is a function. There is a unique function $f : \mathbb{N} \to \mathbb{Z}$ that satisfies

$$f(n) = \begin{cases} b, & n = 0 \\ g(n, f(n-1)), & n > 0 \end{cases}.$$

PROOF (EXISTENCE, WELL-ORDERING): Suppose f does not exist. Then there is some smallest $n \in \mathbb{N}$ for which $f(n)$ is not defined. By the definition above, $n \neq 0$, since $f(0) = b$ is unambiguously defined. That means that $n > 0$, and (by the assumption that n is the smallest natural number for which $f(n)$ is undefined), $f(n-1)$ is defined. But then $f(n) = g(n, f(n-1))$ is defined (by assumption), contradicting the claim that $f(n)$ is not defined. Thus the assumption that $f(n)$ is not defined is false. QED.

PROOF (UNIQUENESS, WELL-ORDERING): Suppose $f(n)$ and $f'(n)$ both satisfy the equation, and yet $f(n) \neq f'(n)$ for some $n \in \mathbb{N}$. Then there is some smallest natural number, $\widehat{n}$, for which $f(\widehat{n}) \neq f'(\widehat{n})$. Clearly $\widehat{n} \neq 0$, since $f(0) = b = f'(0)$, by definition. Since $\widehat{n} > 0$, by the choice of $\widehat{n}$ you know that $f'(\widehat{n}-1) = f(\widehat{n}-1)$, so

$$f(\widehat{n}) = g(\widehat{n}, f(\widehat{n}-1)) = g(\widehat{n}, f'(\widehat{n}-1)) = f'(\widehat{n})$$

contradicting the assumption that $f(\widehat{n}) \neq f'(\widehat{n})$. Thus the assumption that there is some natural number with $f(n) \neq f'(n)$ is false. QED.

4. Here is a recursive definition of a function that takes a pair of natural numbers as arguments.

$$\forall m \in \mathbb{N}, 0 \leq n \leq m \qquad PT(m,n) = \begin{cases} 1, & n = 0 \\ 1, & n = m \\ PT(m-1, n-1) + PT(m-1, n), & 0 < n < m \\ \text{undefined}, & \text{otherwise} \end{cases}$$

Verify the following for $m \in \{0, 1, 2, 3\}$, and then prove the following for all $m \in \mathbb{N}$:

(a) $\sum_{n=0}^{m} PT(m,n) = 2^m$.

For $m = 0$, the claim says that $PT(0,0) = 2^0$ or $1 = 1$, which is true. For $m = 1$, the claim says that $PT(1,0) + PT(1,1) = 2^1$, or $1 + 1 = 2$, which is true. For $m = 2$, the claim says that $PT(2,0) + PT(2,1) + PT(2,2) = 2^2$, or $1 + 2 + 1 = 4$, which is true. For $m = 3$, the claim says that $PT(3,0) + PT(3,1) + PT(3,2) + PT(3,3) = 2^3$, or $1 + 3 + 3 + 1 = 8$, which is true.

PROOF (SIMPLE INDUCTION ON m): For $m = 0$ the base case has been verified in the previous paragraph.

INDUCTION STEP: Assume that the claim holds for some arbitrary natural number m. To show that this implies that the claim holds for $m+1$, I break up the sum $\sum_{n=0}^{m+1} PT(m,n)$ to separate out the first and last terms, and then use the induction hypothesis (IH) as follows

$$\begin{aligned}
\sum_{n=0}^{m+1} PT(m+1,n) &= 1 + \left[\sum_{n=1}^{m} PT(m+1,n)\right] + 1 \\
\text{[definition of } PT(m+1,n)] &= 1 + \left[\sum_{n=1}^{m} PT(m,n-1) + PT(m,n)\right] + 1 \\
[\text{let } \widehat{n} = n-1] &= PT(m,m) + \sum_{\widehat{n}=0}^{m-1} PT(m,\widehat{n}) + \sum_{n=1}^{m} PT(m,n) + PT(m,0) \\
[\text{by IH}] &= \sum_{\widehat{n}=0}^{m} PT(m,\widehat{n}) + \sum_{n=0}^{m} PT(m,n) = 2 \times 2^m = 2^{m+1}.
\end{aligned}$$

Thus the claim holds for $m+1$ whenever it holds for m, as wanted.
I conclude that the claim holds for every $m \in \mathbb{N}$. QED.

(b) If $0 \le n \le m$, then $PT(m,n) = PT(m, m-n)$

FOR $m = 0$: the claim says that $PT(0,0) = PT(0,0)$, which is clearly true.

FOR $m = 1$: the claim says that $PT(1,0) = PT(1,1)$, or $1 = 1$, which is true.

FOR $m = 2$: the claim says that $PT(2,0) = PT(2,2)$, and that $PT(2,1) = PT(2,1)$, or $1 = 1$ and $2 = 2$, which is true.

FOR $m = 3$: the claim says that $PT(3,0) = PT(3,3)$, and $PT(3,1) = PT(3,2)$, or $1 = 1$ and $3 = 3$, which is true.

PROOF (SIMPLE INDUCTION ON m): For $m = 0$, the base case has been verified above.

INDUCTION STEP: Assume that the claim holds for some arbitrary natural number m. In order to show that this implies that the claim holds for $m+1$ there are three cases to consider, depending on which n in the range $0 \le n \le m+1$ you consider:

i. If $n = 0$, then $PT(m+1,n) = 1 = PT(m+1, m+1-n) = 1$, so the claim holds in this case.

ii. If $n = m+1$, then $PT(m+1, m+1) = 1 = PT(m+1, m+1-n) = 1$, so the claim holds in this case.

iii. If $0 < n < m+1$, then you also have $0 < m+1-n < m+1$, and (by definition of $PT(m+1,n)$)

$$\begin{aligned}
PT(m+1,n) &= PT(m,n-1) + PT(m,n) \\
[\text{by IH}] &= PT(m, m-[n-1]) + PT(m, m-n) \\
&= PT(m, m+1-n) + PT(m, m+1-n-1) \\
\text{[by definition of } PT(m+1, m+1-n)] &= PT(m+1, m+1-n).
\end{aligned}$$

Thus, whenever the claim holds for m it also holds for $m+1$, as wanted.
Conclude that the claim holds for all $m \in \mathbb{N}$. QED.